this type. KHSO$_4$ affords a notable example where three curves of this type meet in a triple point. For a liquid, $\Delta v$ always decreases with increasing temperature on either a rising or a falling curve. On the rising transition curves there are 37 cases of normal variation of $\Delta v$ and 5 of abnormal variation; on the falling curves 8 normal and 8 abnormal cases.

The relative compressibility, thermal expansion, and specific heat of neighboring phases is significant. It is natural to expect that the phase of smaller volume will have the smaller compressibility and thermal expansion, and that the phase stable at the higher temperature will have the higher specific heat. If we call this behavior 'normal,' then on rising curves we find 9 cases of normal and 11 of abnormal compressibility, and on falling curves 1 normal and 7 abnormal. The expansion shows 5 normal and 7 abnormal cases on rising curves and 2 normal and 4 abnormal on falling curves. $C_p$ is normal in 5 cases and abnormal in 7 cases on rising curves, and normal in 6 cases and abnormal in 1 on falling curves. The fact of abnormal $C_p$ is of considerable significance from the point of view of the quantum hypothesis. It means (if we may apply the same considerations to $C_p$ as to $C_v$, which is usually done) that the specific heat curves of the two modifications cannot be of the same character, but that somewhere between the transition point and absolute zero the one which is lower at the transition point must cross and lie above the other.

In addition to the substances enumerated above, about 100 others have been examined without finding other forms.


ON ISOTHERMALLY CONJUGATE NETS OF SPACE CURVES

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Presented to the Academy, August 10, 1915

Bianchi$^1$ has called a parametric net of curves on a surface isothermally conjugate if, when the surface is referred to these curves, the second fundamental form, $D \, du^2 + 2D' \, du \, dv + D'' \, dv^2$, may by a transformation $\overline{u} = U(u), \overline{v} = V(v)$ be made to take on the same shape as does the first fundamental form when the parametric net is isothermal; i.e., the parametric net is isothermally conjugate if $D' = 0$, $D = D''$. These nets have lately attained increased importance, so that Wilczynski's recent geometric interpretation$^2$ of Bianchi's condi-
tion is of great interest. In the present note, we propose to give a new and simple geometric characterization of isothermally conjugate nets which is entirely different from Wilczynski's.

Let \( y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)} \), be the homogeneous coordinates of a point in space, and let the four functions

\[
y^{(k)} = f^{(k)}(u, v) \quad (k = 1, 2, 3, 4)
\]

define a surface \( S_y \) on which the curves \( u = \text{const.}, v = \text{const.} \) form a conjugate net. Then the \( y^{(k)} \)'s satisfy a completely integrable system of two partial differential equations of the form

\[
\begin{align*}
y_{uu} &= ay_u + by_v + cy + dy, \\
y_{uv} &= b'y_u + c'y_v + d'y.
\end{align*}
\]

The second of these is of the familiar Laplace type, characteristic of conjugate nets; the first shows that the conjugate net defined by equations (1) is isothermally conjugate if and only if

\[
\frac{\partial^2}{\partial u \partial v} \log a = 0.
\]

The coefficients in equations (2) are not arbitrary, but are subjected to certain integrability conditions. One of the relations yielded by these conditions is that

\[
\frac{\partial}{\partial v} (b + 2c') = \frac{\partial}{\partial u} \left( 2b' - \frac{c}{a} - \frac{\partial}{\partial v} \log a \right),
\]

or

\[
b' + 2c' = 2b' - \left( \frac{c}{a} \right) - \frac{\partial^2}{\partial u \partial v} \log a.
\]

The minus first and first Laplace transforms of the point \( y \) are respectively

\[
\rho = y_u - c'y, \quad \sigma = y_v - b'y,
\]

which represent covariant points on the tangents at \( y \) to the curves of the net passing through \( y \). The surface \( S_\rho \) is the second focal sheet of the congruence of tangents to the curves \( v = \text{const.} \) on \( S_y \), and \( S_\sigma \) is the second focal sheet of the congruence of tangents to the curves \( u = \text{const.} \) on \( S_y \). Let us, with Wilczynski,\(^2\) call the line \( \rho \sigma \) corresponding to the point \( y \) the ray of the point \( y \), and the totality of rays, which form a congruence, the ray congruence.

The osculating planes of the two curves \( u = \text{const.} \) and \( v = \text{const.} \) at a point \( y \) meet in a line which passes through \( y \) and which Wilczynski
calls the *axis* of the point \( y \). The totality of axes, which correspond to all the points \( y \) of the surface \( S_y \), constitute a congruence, the *axis congruence*.

We may write the first of equations (2) in the form

\[
y_{uv} - by_u - dy = ay_v + cy_v.
\]

The left-hand member represents a point in the osculating plane to the curve \( v = \text{const.} \), and the right-hand member a point in the osculating plane to the curve \( u = \text{const.} \), at \( y \). Therefore, since the coordinates are homogeneous, the point

\[
z = y_v + \frac{c}{a} y_v
\]

lies on the line of intersection of the two osculating planes, and the line \( yz \) is the axis of the point \( y \).

We may determine the developables of the axis congruence as follows. If the point \( y \) moves to the point \( y + dy \), the point \( z \) moves to \( z + ds \), where \( dy = y_u du + y_v dv \) and \( ds = z_u du + z_v dv \). We wish the line \( yz \) to generate a developable. This will happen if and only if the four points \( y, z, y + dy, z + ds \) lie in a plane, or what is the same thing, if the points \( y, z, y_u du + y_v dv, z_u du + z_v dv \) are coplanar. We have on differentiation of equations (2)

\[
y_{uuv} = a^{(12)} y_{uv} + \beta^{(12)} y_u + \gamma^{(12)} y_v + \delta^{(12)} y_v,
\]

\[
y_{vv} = a^{(03)} y_v + \beta^{(03)} y_u + \gamma^{(03)} y_v + \delta^{(03)} y_v,
\]

where in particular

\[
\begin{align*}
\alpha^{(12)} &= b', \quad \beta^{(12)} = b'' + b', \quad \gamma^{(12)} = b'c' + c' + d', \\
\alpha^{(03)} &= b' - \frac{c}{a} \frac{\partial}{\partial v} \log a, \quad \beta^{(03)} = \frac{1}{a} (b'c' + b' - b_v + d'), \\
\gamma^{(03)} &= \frac{1}{a} \left[ b'c + c'(c' - b) + c'_v - c_v - d \right],
\end{align*}
\]

so that on using these and equations (2) we find

\[
z_u = y_{uuv} + \frac{c}{a} y_{uv} + \left( \frac{c}{a} \right)_u y_v
\]

\[
= c'y_v + \left( b^{(12)} + \frac{b'c}{a} \right) y_u + \left[ \gamma^{(12)} + \frac{c}{a} + \left( \frac{c}{a} \right)_u \right] y_v + (c y_v),
\]

\[
z_v = y_{vv} + \frac{c}{a} y_v + \left( \frac{c}{a} \right)_v y_v
\]

\[
= \left( b' - \frac{\partial}{\partial v} \log a \right) y_v + \beta^{(03)} y_u + \left[ \gamma^{(03)} + \left( \frac{c}{a} \right) \right] y_v + (c y_v),
\]
in which the coefficients of \( y \) do not concern us. Consequently,

\[
z_u du + z_v dv = \left[ c'du + \left( b' - \frac{\partial}{\partial v} \log a \right) dv \right] y_u + \left[ \left( \beta^{(12)} + \frac{b'c}{a} \right) du + \beta^{(03)} dv \right] y_u + \left[ \left( \gamma^{(12)} + \frac{c'}{a} \right) du + \left( \gamma^{(03)} + \frac{c}{a} \right) dv \right] y_v + \left( \right) y.
\]

Now, if the points \( y, z, dy, dz \) are to be coplanar, the determinant of the coefficients of \( y, y, y, \) in the expressions for \( z, dy, dz \) must vanish; on expansion this determinant yields the quadratic in \( du: dv, \)

\[
a \left[ \gamma^{(12)} + \left( \frac{c}{a} \right) \right] du^2 - \mathcal{D} du dv - a \beta^{(03)} dv^2 = 0,
\]

where, on using (4), we find

\[
\mathcal{D} = d + ab'^2 - c'^2 + b'c + bc' + ab' - c'.
\]

The quadratic (6) determines the direction in which \( y \) must move, in order that the axis \( yz \) may trace out a developable; there are two such directions at each point of \( S_y. \) We may regard (6) as a differential equation defining a net of curves on \( S_y \) having the property that if the point \( y \) traces out a curve of this net, the corresponding axis generates a developable surface. We call the two curves of the net which pass through the point \( y \) the \textit{axis curves} of the point \( y. \)

In like manner, we may determine the developables of the ray congruence, i.e., the net of curves on \( S_y \) having the property, that if the point \( y \) traces out a curve of the net, the corresponding ray traces out a developable of the ray congruence. The differential equation defining this net of curves, which we call the \textit{ray curves}, is without difficulty found to be

\[
a \mathcal{H} du^2 - \mathcal{D} du dv - \mathcal{K} dv^2 = 0,
\]

where \( \mathcal{D} \) is given by (7), and

\[
\mathcal{H} = d' + b'c' - b'_v, \quad \mathcal{K} = d' + b'c' - c'_v
\]

are the Laplace-Darboux invariants of the given conjugate net.

If we use (9), we find from (5) that

\[
a \beta^{(03)} = \mathcal{H} + 2b'_u - b_v, \quad \gamma^{(12)} + \left( \frac{c}{a} \right) = \mathcal{K} + 2c'_v + \left( \frac{c}{a} \right),
\]

the latter of which becomes, on use of (4),

\[
\gamma^{(12)} + \left( \frac{c}{a} \right) = \mathcal{K} + 2b'_v - b_v - \frac{\partial^2}{\partial u \partial v} \log a.
\]
The differential equation (6) of the axis curves may therefore be written
\[ a \left[ K + 2b'u - b_v - \frac{\partial^2 \log a}{\partial u \partial v} \right] du^2 - D \, du \, dv - (H + 2b'u - b_v) \, dv^2 = 0. \] (10)

The differential equation of the asymptotic curves is
\[ a du^2 + dv^2 = 0 \] (11)

The pair of asymptotic tangents at \( y \) is of course harmonically separated by the tangents to the curves of our conjugate net. The differential equation
\[ a du^2 - dv^2 = 0 \] (12)
defines a new net of curves. It evidently has the property, that the tangents to the two curves of the net at the point \( y \) separate harmonically both the pair of asymptotic tangents and the tangents to the two curves of our conjugate net. It is moreover the only net which has this property; since it also is a conjugate net, we call it the associate conjugate net.

We shall define another net of curves which will be of importance in our geometric interpretation. The quadratic
\[ a H \, du^2 + D \, du \, dv - K \, dv^2 = 0 \] (13)
has for its roots the negatives of the roots of (8). It therefore defines a net such that the tangents to the two curves thereof at the point \( y \) are the harmonic conjugates of the two ray tangents (the tangents to the ray curves) with respect to the original conjugate tangents (the tangents to the curves of the original conjugate net). For convenience, let us call the curves defined by (13) the anti-ray curves, and the two tangents to the anti-ray curves at the point \( y \) the anti-ray tangents of the point \( y \).

Let us now fix our attention upon a point \( y \) of the surface \( S_y \), and let us regard equations (10), (12), and (13) as binary quadratics whose roots give respectively the pairs of axis tangents, associated conjugate tangents, and anti-ray tangents of the point \( y \). The Jacobian of the forms (10) and (12) is
\[ a D \, du^2 + 2a \left( H - K + \frac{\partial^2 \log a}{\partial u \partial v} \right) du \, dv + D \, dv^2 = 0, \] (14)
and its roots give the pair of lines through \( y \) which separate harmonically both the pair of axis tangents and the pair of associated conjugate tangents of \( y \). The Jacobian of the forms (12) and (13) is
and defines the pair of lines through $y$ which separate harmonically both the pair of anti-ray tangents and the pair of associate conjugate tangents of the point $y$.

The two Jacobians (14) and (15) coincide if and only if

$$\frac{\partial^2}{\partial u \partial v} \log a = 0,$$

(3 bis)
i.e., if and only if the original conjugate net is isothermally conjugate.

We may state our result as follows:

A necessary and sufficient condition that a conjugate net of curves on a surface be isothermally conjugate is that at each point of the surface the pair of axis tangents, the pair of associate conjugate tangents, and the pair of anti-ray tangents be pairs of the same involution.

By means of the various nets of curves defined in the course of the above interpretation, we have been enabled to deduce a number of properties of isothermally conjugate nets. We have included this more extended discussion in a longer paper, which is a sequel to the one on conjugate nets to which reference has already been made.

4. Ibid., end of §3.


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Presented to the Academy, August 18, 1915

It is very generally considered by all except those who have paid special attention to the subject, that the number of red corpuscles per unit volume of blood is, in the normal individual, a fairly fixed quantity subject to gradual change only. A more careful study shows however that this number is subject to very rapid and great changes, and instead of being constant, that it is continually changing under physiological conditions.

Questions naturally arise as to what factors will cause a change in