For testing the importance of the lymphoid tissue in natural immunity we can only compare the percentage of takes in X-rayed and normal animals inoculated with the same cancer. This has been done with a variety of different cancers and a large series of animals. The average number of takes in the X-rayed animals was 94%, while in the untreated animals only 32% of those inoculated grew the cancer. This shows a very considerable destruction of the natural immunity accompanying a destruction of the lymphocytes.

To summarize, we have shown that a marked increase in the circulating lymphocytes occurs after cancer inoculation in mice with either a natural or induced immunity. When this lymphoid reaction is prevented by a previous destruction of the lymphoid tissue with X-ray the immune states are destroyed. Hence it would seem fair to conclude that the lymphocyte is a necessary factor in cancer immunity.

SOME THEOREMS CONNECTED WITH IRRATIONAL NUMBERS

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As is well known to those who have investigated the fields of celestial mechanics, the series which arise there from the integration of the equations of motion involve factors of the form \((i - j\gamma)\) in the denominators of the coefficients, where \(i\) and \(j\) run over the entire series of positive integers and \(\gamma\) is a positive number which may be rational or irrational. Previous to the time when Poincaré had shown the existence and construction of periodic solutions (in which \(\gamma\) is always rational) it had been the custom for the astronomers to regard \(\gamma\) as irrational since with this hypothesis the factors \((i - j\gamma)\) never vanish and consequently non-periodic terms did not arise in the solutions. The presence of these factors in the denominators naturally led to very grave doubts as to the convergence of these series since there are infinitely many such factors which are smaller than any assigned limit, and the convergence has never been proved.

Considerations of this nature have led me recently to examine the convergence of simple types of power series in which this phenomenon occurs, and it has been found that the series \(\sum a_i/(i - j\gamma) x^i y^j\) has precisely the same domain of convergence as the series \(\sum a_i x^i y^j\), provided \(\gamma\) is a positive irrational number which satisfies a rather mild condition, which is stated below.
Further investigations in this field have resulted in the three following theorems:

**Theorem I.** If \( \gamma \) is any positive number, rational or irrational, and if \( \frac{p_n}{n} \) is a rational fraction such that \( |p_n - n\gamma| \leq \frac{1}{2} \), and if

\[
A_n = \frac{1}{n} \sum_{k=1}^{n} (p_k - k\gamma)
\]

is the arithmetic mean of the first \( n \) of the quantities \( (p_k - k\gamma) \), signs considered, then the limit of \( A_n \), as \( n \) increases without limit, is zero.

If \( \gamma = \frac{p}{q} \) is rational with an even denominator \( q \) then for certain values of \( \kappa \) there are two integers \( p_\kappa \) which differ by unity such that \( |p_\kappa - k\gamma| = \frac{1}{2}; \) for one, the value is \( +\frac{1}{2} \) and for the other it is \( -\frac{1}{2} \). It is supposed that such terms are taken alternately \( +\frac{1}{2} \) and \( -\frac{1}{2} \).

**Theorem II.** If \( \gamma \) is a positive number, and if \( \frac{p_n}{n} \) is a rational fraction such that \( |p_n - n\gamma| \leq \frac{1}{2} \), and if

\[
A_n = \frac{1}{n} \sum_{k=1}^{n} |p_k - k\gamma|
\]

is the arithmetic mean of the first \( n \) of the quantities \( |p_k - k\gamma| \), signs discarded, then the limit of \( A_n \), as \( n \) increases without limit, is \( \frac{1}{2} \) if \( \gamma \) is irrational or rational with an even denominator; but if \( \gamma \) is rational with an odd denominator, \( \gamma = \frac{p}{q} \), then the limit of \( A_n \) is \( (q^2 - 1)/4q^2 \).

**Theorem III.** If \( \gamma \) is a positive number, and if \( \frac{p_n}{n} \) is a rational fraction such that \( |p_n - n\gamma| \leq \frac{1}{2} \), and if

\[
G_n = \left( \prod_{k=1}^{n} |p_k - k\gamma| \right)^{\frac{1}{n}}
\]

is the geometric mean of the first \( n \) of the quantities \( |p_k - k\gamma| \), then the limit of \( G_n \), as \( n \) increases without limit, is equal to \( 1/(2e) \) where \( e = 2.71828 \ldots \) is the naperian base, if \( \gamma \) is an irrational number which satisfies the condition

\[
a_{n+1} \leq M q_n (q_n + 1) \ldots (q_n + s),
\]

where \( \gamma \), expressed as a simple continued fraction, is

\[
\gamma = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \ldots}}},
\]

\( q_n \) is the denominator of the \( n \text{th} \) principal convergent, and \( s \) any assigned positive integer independent of \( n \). If \( \gamma \) is an irrational number which does not satisfy this condition then \( G_n \) for large values of \( n \) oscillates between zero and \( 1/(2e) \).