of the surfaces $z_i$ with the property that a curve $y^i$ intersects a surface $y_j$ once or zero times, modulo $\tau$, according as $i$ is equal to or different from $j$. In other words, the intersection of $y^i$ with $y_j$ is determined by the coefficient of $y^i y_j$ in the bilinear form

$$y^i y_j \pmod{\tau}$$

Let $\pi^{ij}$ denote the number of times, modulo $\tau$, that a curve $y^i$ cuts the surface $z_j$ bounded by $y^j$. The numbers $\pi^{ij}$ are the components of a contravariant tensor of the second order, this time with respect to the variables $y_i$. In the general case, there will be a tensor $\pi^{ij}$ corresponding to each value of the coefficients of torsion.

The tensors $\sigma_{ijk}$ and $\rho_{ijk}$ are not derivable from the Betti numbers and coefficients of torsion of a manifold, as may be seen by comparing the two manifolds given in my previous note. By means of the tensors $\pi^{ij}$, it is even possible to distinguish between manifolds with the same group. There are, for instance, two distinct manifolds of group 5 such that the Heegaard diagram of each may be formed on an anchor ring, but such that the characteristic curve of one diagram is $ab^5$, of the other $a^2b^5$. For the first manifold, the possible values of the tensor $\pi^{11}$ are 1 and 4, modulo 5, depending upon the choice of the fundamental surface $y_{11}$; for the second manifold, the possible values are 2 and 3.

Evidently, the above discussion may be extended from three to $n$ dimensions.

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**ON CERTAIN NEW TOPOLOGICAL INVARIANTS OF A MANIFOLD**

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In an $n$-dimensional manifold $M_n$ let

$$(C^k) \qquad C^k_1, C^k_2, \ldots, C^k_k$$

$$(C^l) \qquad C^l_1, C^l_2, \ldots, C^l_l$$

$$(C^m) \qquad C^m_1, C^m_2, \ldots, C^m_m$$

be complete sets of linearly independent non-bounding circuits of dimensionalities $k$, $l$, and $m$ respectively, such that

$$k + l > n, \quad m = k + l - n.$$ 

Then, any $k$-dimensional circuit $C_k$ is homologous to a linear combination of the circuits $(C^k_1)$
\[ C_k \sim \alpha_1 C_k^1 + \alpha_2 C_k^2 + \ldots + \alpha_p C_k^p, \]

where the numbers \( \alpha_1, \alpha_2, \ldots, \alpha_p \) are integers. Similarly, every \( l \)-circuit \( C_l \) is homologous to a linear combination of the circuits \( (C_l^1) \), and every \( m \)-circuit \( C_m \) to a linear combination of the circuits \( (C_m^i) \). Moreover, two circuits \( C_k^i \) and \( C_l^j \) intersect in a circuit \( C_m^i \), so that we may write

\[ C_m = C_k^i \cdot C_l^j \sim \alpha_1 C_m^1 + \alpha_2 C_m^2 + \alpha_p C_m^p. \]

Construct a rectangular matrix \( J_{pq} \) of \( p \) rows and \( q \) columns, one row for each \( k \)-circuit of \( (C_k^i) \), one column for each \( l \)-circuit of \( (C_l^j) \), and let the element in the \( i \)-th row and the \( j \)-th column be the linear combination of the \( m \)-circuits \( (C_m^i) \) homologous to the intersection of \( C_k^i \) with \( C_l^j \). The matrix \( J_{pq} \) may be modified by making linear transformations with integer coefficients and determinants \( \pm 1 \) on the rows of the matrix, on the columns of the matrix, and on the symbols \( C_m^i \) occurring in the elements of the matrix. It is easy to prove that the group of all matrices so obtained by these linear transformations is a topological invariant of the manifold \( M_n \), or, in other words, that if \( M_n \) and \( M'_n \) be homeomorphic manifolds, it must be possible to transform the matrix \( J_{pq} \) of \( M_n \) into the corresponding matrix \( J_{pq}' \) of \( M'_n \) by elementary transformations of the type just described.

The matrices \( J_{pq} \) are not dependent on the Betti numbers and coefficients of torsion of Poincaré, as a comparison of the following two examples will show:

First Example.—Consider the 3-dimensional manifold obtained by removing from spherical 3-space the interiors of six spheres, and deforming the residual space so as to match the surfaces of the spheres together pointwise in pairs, in such a manner as to obtain an orientable closed manifold \( M_n \). The Betti numbers of the manifold are obviously

\[ (P_1 - 1) = (P_2 - 1) = 3, \]

and there is no torsion. Moreover, the matrix \( J_{22} \) of intersections of the 2-circuits with one another is

\[
\begin{array}{ccc}
S_1^1 & S_1^2 & S_1^3 \\
S_2^1 & 0 & 0 & 0 \\
S_2^2 & 0 & 0 & 0 \\
S_3^3 & 0 & 0 & 0 \\
\end{array}
\]

as may be seen by taking each pair of matched spheres as a fundamental 2-circuit.

Second Example.—Consider the 3-dimensional manifold \( M'_n \) determined by the points and interior points of a cube in 3-space when the points of opposite faces of the cube are matched with one another as they would be if one face were carried into its opposite by a translation. Again, we have

\[ (P_1 - 1) = (P_2 - 1) = 3, \]
and there is no torsion. Each pair of opposite faces determines a non-bounding 2-circuit, each set of parallel edges, a non-bounding 1-circuit. In place of $J_{22}$, however; we obtain the matrix

$$
\begin{pmatrix}
S_1^1 & S_1^2 & S_1^3 \\
S_2^1 & 0 & S_1^2 \\
-S_1^2 & 0 & S_1^1 \\
S_2^3 & S_1^2 & -S_1^1
\end{pmatrix}
$$

This matrix is evidently not reducible to $J_{22}$, since no linear combination of the three rows with integer coefficients not all zero can be made to vanish.

A fuller report on the above matrices and on others obtained by studying the intersection of more than two systems of circuits will be made elsewhere. In particular I shall prove that the matrix $J_{44}$ may be replaced by a 3-dimensional array with integer elements.

The possibility of finding some such invariants as the ones obtained in this note was suggested a long time ago by Professor Veblen.

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**CONDITION THAT AN ELECTRON DESCRIBE A GEODESIC**

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Consider a Riemann space, whose linear element is written in the form

$$
\text{d}s^2 = g_{ij} \text{d}x^i \text{d}x^j
$$

The equations of motion of an electron\(^1\) are

$$
\mu g_{ij} \left( \frac{\text{d}x^j}{\text{d}s} + \Gamma_{\alpha \beta}^i \frac{\text{d}x^\alpha}{\text{d}s} \frac{\text{d}x^\beta}{\text{d}s} \right) = (S_i^k) k
$$

where $\mu$ is the density and

$$
\Gamma_{\alpha \beta}^i = \frac{1}{2} \left( \frac{\partial g_{\alpha k}}{\partial x^\beta} + \frac{\partial g_{\beta k}}{\partial x^\alpha} - \frac{\partial g_{\alpha \beta}}{\partial x^k} \right) g^{ik}
$$

If Maxwell's equations are

$$
F_{\mu \nu} = \frac{\partial k \mu}{\partial x^\nu} - \frac{\partial k \nu}{\partial x^\mu}
$$

and

$$
F_{\mu \nu} = J^\mu
$$

where

$$
J^\mu = \rho \left( \frac{\text{d}x^1}{\text{d}s}, \frac{\text{d}x^2}{\text{d}s}, \frac{\text{d}x^3}{\text{d}s}, \frac{\text{d}x^4}{\text{d}s} \right)
$$