Partitions with difference conditions and Alder’s conjecture

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In 1956, Alder conjectured that the number of partitions of \( n \) into parts differing by at least \( d \) is equal to the number of partitions of \( n \) into parts with minimal difference \( 3 \) between parts and no consecutive multiples of \( 3 \) for \( d \geq 4 \). In 1971, Andrews proved that the conjecture holds for \( d = 2^r - 1, r \geq 4 \). We sketch a proof of the conjecture for all \( d \geq 32 \).

1. Introduction

The well-known Rogers–Ramanujan identities may be stated partition-theoretically as follows. If \( c = 1 \) or \( 2 \), then the number of partitions of \( n \) into parts \( = c \) (mod 5) equals the number of partitions of \( n \) into parts \( \geq c \) with minimal difference 2 between parts. In 1926, I. Schur (1) proved that the number of partitions of \( n \) into distinct parts equals the number of partitions of \( n \) into parts \( = \pm 1 \) (mod 6).

In 1956, H. L. Alder (2) posed the following problems. Let \( q_d(n) \) be the number of partitions of \( n \) into parts differing by at least \( d \); let \( Q_d(n) \) be the number of partitions of \( n \) into parts \( = \pm 1 \) (mod \( d + 3 \)). (a) Is \( \Delta_d(n) \) nonnegative for all positive \( d \) and \( n \)? It is known that \( \Delta_d(n) = 0 \) for \( n \) greater than or equal to 0 by Euler’s identity, that the number of partitions of \( n \) into distinct parts equals that of partitions of \( n \) into odd parts, that \( \Delta_d(n) = 0 \) for \( n \) greater than or equal to 0 by Schur’s theorem, which states that \( \Delta_d(n) \) equals the number of partitions of \( n \) into odd parts differing by at least \( d \) that contain at least one pair of consecutive multiples of \( 3 \). (b) If problem a is true, can \( \Delta_d(n) \) be characterized as the number of partitions of \( n \) for a certain type, as is the case for \( d = 3 \)?

In 1971, G. E. Andrews (3) gave some partial answers to problem a.

Theorem 1.1 (ref. 3). For any \( d \geq 4 \), \( \lim_{n \to \infty} \Delta_d(n) = \infty \).

Theorem 1.2 (ref. 3). If \( d = 2^r - 1, r \geq 4 \), then \( \Delta_d(n) \geq 0 \) for all \( n \).

To prove Theorem 1.2, Andrews studied the set of partitions of \( n \) into distinct parts \( = 2 \) (mod \( d \)) for \( 0 \leq i < r \), the size of which is greater than or equal to \( Q_d(n) \) for any \( n \), and he succeeded in finding a set of partitions with difference conditions between parts that are complicated but much stronger than the difference condition for \( q_d(n) \). He then proved that partitions into distinct parts \( = 2 \) (mod \( d \)) for \( 0 \leq i < r \) and partitions with the difference conditions are equinumerous. As a result, he was able to prove Alder’s conjecture in these particular cases.

By generalizing Andrews’ methods and constructing an injection, we are able to prove that Alder’s conjecture holds for \( d = 7 \) and \( d \geq 32 \) (unpublished work). Therefore, Alder’s conjecture still remains unresolved for \( 4 \leq d \leq 30 \) and \( d \neq 7, 15 \).

In Section 2, we will give an outline of the proof of the case when \( d = 2^r - 1, r \geq 4 \) by Andrews, and in Section 3, we will sketch the proof of the case when \( d \geq 32 \).

In the sequel, we assume that \( \left| q \right| < 1 \) and use the customary notation for \( q \)-series

\[
(a)_0 := (a; q)_0 := 1,
\]

\[
(a)_\infty := (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k),
\]

\[
(a)_n := (a; q)_n := \frac{(a; q)_\infty}{(aq^n; q)_\infty} \quad \text{for any } n.
\]

2. The Case When \( d = 2^r - 1 \)

From the definitions of \( q_d(n) \) and \( Q_d(n) \), the generating functions for \( q_d(n) \) and \( Q_d(n) \) are, respectively,

\[
\sum_{n=0}^{\infty} q_d(n)q^n = \sum_{n=0}^{\infty} q_d^{(2)}(n)q^n;
\]

\[
\sum_{n=0}^{\infty} Q_d(n)q^n = \frac{1}{(q; q^d)^{n}(q^d+2; q^d)^{n}(q^d+4; q^d)^{n} \cdots (q^{d+2r-2}; q^d)^{n}}
\]

(see ref. 4, chapter 1).

For a given \( d = 2^r - 1 \), define

\[
f_d(q) = \sum_{n=0}^{\infty} L_d(n)q^n = (-q^d; q^d)_\infty(-q^{2d}; q^d)_\infty \cdots (-q^{2^{r-1}d}; q^d)_\infty.
\]

Then

\[
f_d(q) = \frac{1}{(q; q^2d)(q^{d+2}; q^2d)(q^{d+4}; q^2d) \cdots (q^{d+2r-2}; q^2d)}.
\]

Let

\[A_d = \{ m \mid m = 2^i \pmod{d}, 0 \leq i \leq r - 1 \}\]

and

\[A'_d = \{ m \mid m = i \pmod{d}, 1 \leq i \leq 2^r - 1 \} .\]

Let \( \beta_d(m) \) denote the least positive residue of \( m \) modulo \( d \). For \( m \in A_d \), let \( b(m) \) be the number of terms appearing in the binary representation of \( m \), and let \( v(m) \) denote the least \( 2^i \) in this representation. We need a theorem of Andrews (S), which we state without proof in the following theorem.

Theorem 2.1. Let \( D(A_d; n) \) denote the number of partitions of \( n \) into distinct parts taken from \( A_d \), and let \( E(A'_d; n) \) denote the number of partitions of \( n \) into parts taken from \( A'_d \) of the form \( n = \lambda_1 + \lambda_2 + \cdots + \lambda_r \), such that

\[
\lambda_i - \lambda_{i+1} \geq d-b(\beta_d(\lambda_{i+1})) + v(\beta_d(\lambda_{i+1})) - \beta_d(\lambda_{i+1}). \quad \text{[2.1]}
\]

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Theorem 2.2. Let $S = \{a_i\}_{i=1}^\infty$ and $T = \{b_i\}_{i=1}^\infty$ be two strictly increasing sequences of positive integers such that $b_1 = 1$ and $a_i \geq b_i$ for all $i$. Let $p(S; n)$ and $p(T; n)$ denote the numbers of partitions of $n$ into parts taken from $S$ and $T$, respectively. Then, for $n \geq 1$, 

$$p(T; n) \geq p(S; n).$$

By comparing parts arising in partitions counted by $Q_d(n)$ and $L_d(n)$, we see that $L_d(n) = D(A_d; n)$ and $q_d(n) = E(A_d; n)$, it follows from Theorem 2.1 that $q_d(n) \geq L_d(n)$. Therefore, we have shown that $q_d(n) \geq Q_d(n)$ for $d = 2^r - 1$, $r \geq 4$.

3. The Case When $d \neq 2^r - 1$

We denote the coefficient of $q^n$ in an infinite series $s(q)$ by $[q^n]s(q)$. For a given $d$, uniquely define the integer $r$ by 

$$2^r < d + 1 < 2^{r+1},$$

and let $L_d(n)$ be the number of partitions of $n$ into distinct parts $= 1, 2, 4, \ldots, 2^r - 1 \pmod{d}$. Then the generating function $f_d(q)$ for $L_d(n)$ is 

$$f_d(q) = \sum_{n=0}^{\infty} L_d(n)q^n = (-q, -q^2, -q^4, \ldots, -q^{2^r}; q^d).$$

By examining the generating function $f_d(q)$, we find from Theorem 2.2 that for $d \geq 32$, 

$$L_d(n) + L_d(n - 2^r) \geq Q_d(n), \quad n \geq 1,$$

where $L_d(n) = 0$ if $m \leq 0$. Thus, we only need to prove that 

$$q_d(n) \geq L_d(n) + L_d(n - 2^r).$$

Let $X(d; n)$ and $Y(d; n)$ be the sets of partitions of $n$ counted by $q_d(n)$ and $L_d(n)$, respectively. Then

$$(X(d; n) \supseteq Y(d; n),$$

because the conditions for $Y(d; n)$ are much stronger than those for $X(d; n)$. Thus, if there exists an injection from $Y(d; n - 2^r)$ to $X(d; n)\setminus Y(d; n)$, then the Alder conjecture holds.

Lemma 3.1. For any $d \geq 32$ not of the form $2^r - 1$, 

$$q_d(n) \geq L_d(n) + L_d(n - 2^r), \quad n \geq 4d + 2^r.$$  

To obtain Lemma 3.1, we construct an injection from $Y(d; n - 2^r)$ to $X(d; n)$ such that the image of a partition in $Y(d; n - 2^r)$ does not satisfy the conditions for $Y(d; n)$. For $n < 4d + 2^r$, we can show that $q_d(n) \geq Q_d(n)$ by merely counting $q_d(n)$ and $Q_d(n)$, respectively. Therefore, from Lemma 3.1, inequality 3.1, and Andrews' result we obtain the following theorem.

Theorem 3.2. For $d \geq 31$, 

$$q_d(n) \geq Q_d(n), \quad n \geq 1.$$  

4. Conclusion

The most intriguing identities in the theory of partitions have been the Rogers–Ramanujan identities. Extremely motivated by these identities, Schur searched for further analogous partition identities. In 1926, Schur (1) proved that the number of partitions of $n$ with minimal difference $3$ between parts and no consecutive multiples of $3$ equals the number of partitions of $n$ into parts $\equiv \pm 1 \pmod{6}$.

Alder’s conjecture has naturally arisen from the Rogers–Ramanujan identities and the Schur identity, and has been open for $\approx 50$ years. The difficulty in dealing with the conjecture is that no set $S$ exists such that the number of partitions of $n$ with parts from $S$ is equal to $q_d(n)$ for $d \geq 3$ (see refs. 6 and 7). Thus finding injections between two sets might not be the most efficient method, but it is the best approach for the conjecture at this point.

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