The rise and fall of a networked society: A formal model

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In a well networked community, there is intense social interaction, and information disseminates briskly and broadly. This is important if the environment is volatile (i.e., keeps changing) and individuals never stop searching for fresh opportunities. Here, we present a simple model that attributes the rise of a dynamic society to the emergence of some key features in its social network. We also explain the apparently paradoxical observation that although such features do not necessarily materialize even under favorable conditions they display a significant resilience to deteriorating conditions. We interpret these findings as a discontinuous phase transition in the network formation process.

There is ample empirical evidence supporting the importance of social networks as the channel through which many socioeconomic phenomena unfold. This idea is particularly apparent, for example, in the way in which economic agents find new opportunities, such as jobs or investments. It has been consistently shown by sociologists and economists alike (1, 2) that personal acquaintances or neighborhood effects play a prominent role in individual search. This, in turn, leads to significant correlations across friends, relatives, or neighbors in a variety of different socioeconomic dimensions. The common thesis proposed to explain this evidence is that, in the presence of economic volatility, the quantity and quality of one’s social links, sometimes referred to as social capital (3, 4), is a key basis for search and adaptability to change.

The study of complex networks has attracted much attention (5–9), but it has been concerned mainly with simple phenomenological models reproducing some stylized facts in either stationary or nonstationary (e.g., growing) contexts. In contrast, sociological (10) and economic literature (11) has traditionally placed emphasis on understanding the main features and implications of stable social structures. Recently, however, much effort has been devoted as well to studying the dynamic forces (essentially, purposeful agent adjustment) that underlie the evolution and formation of networks in stationary social environments (12–15). Here, our objective is to integrate these approaches by proposing a simple model of a society that embodies the following three features: agent interaction, search cum adjustment, and volatility (i.e., random link removal). Individuals are involved in bilateral interaction, as reflected by the prevailing network. Through occasional update, the value of some of the existing links deteriorates and is therefore lost. Over time, this leads to an evolving social network that is always adapting to changing conditions.

The model studied here is a simplification of a more complex model proposed by one of the authors (16) to understand how the network dynamics impinges on strategic behavior. One of the key ingredients of our model is creation of links to friends of friends, a mechanism that was introduced by Vazquez (17) in the context of growing networks. The model is also similar to that proposed in ref. 18 to explain the emergence of the small-world property (5) in social networks. In our context, we find as well that the small-world property arises when the social network is dense, but our focus is quite different. Our aim is to understand how a highly connected network may emerge from social interactions and to develop a comprehensive picture of the network’s macroscopic statistical properties. In particular, we find quite nontrivial clustering properties that appear to play a key role in the dynamics. In contrast with ref. 18, our model does not reproduce a scale-free topology, which is instead typical of growing networks (8) and static random networks with fitness-driven attachment rules (19). Rather we find single-scale networks consistent with the empirical evidence of refs. 20 and 21 on several social networks, giving support to their conjecture that link-constrained dynamics leads to single-scale distribution. Finally, among the vast recent literature on network dynamics, our work also relates to ref. 22 that found a “topological” phase transition in networks and refs. 23–25 that discuss robustness of the network with respect to removal of links or nodes and transition from highly connected to diluted networks in various contexts.

The model may be described as follows. There is a population of $n$ agents involved in a set of bilateral interactions, as specified by the prevailing social network. This network is defined, at any given point in time $t$, by the (undirected) graph $\Gamma(t) = \{N(t)\}$, where $N = \{1, 2, \ldots, n\}$ is the population of nodes (or agents) and $g(t) \subseteq N \times N$ represents the set of links. The social interaction taking place across a link $ij \in g$ between $i$ and $j$ may be conceived as, say, a collaboration that is profitable for both parties.

In any time interval $[t, t + dt]$ any existing link $ij \in g(t)$ vanishes with probability $\lambda dt$. This is interpreted as a random perturbation of the environment, or volatility for short. In addition, with independent probability $\mu dt$ every agent $i$ is given the opportunity of establishing a new link with some other agent $j$, randomly drawn from the population. Links can also be formed through search via friends; every agent $i$, with a probability $\gamma dt$, asks one of his neighbors $j$, randomly chosen, to introduce him to one of $j$’s neighbors, say $k$. If $k$ is not already a neighbor of $i$ the link $ik$ is established. Naturally, nothing occurs if $i$ has no neighbors or $j$ has no other neighbor but $i$.

At a heuristic level, the link formation process can be decomposed into two complementary components. On the one hand, there is the force of volatility that stumps out the value of some preexisting links and thus, in effect, destroys them. On the other hand, there are fresh new opportunities that arise through either global search or communication with neighbors. This 2-fold interpretation of the process makes the role of information clear. The dynamics of network formation can be viewed as a continuous struggle against volatility, with the information arising on new profitable opportunities partially mediated (thus constrained) by the existing network. In the stationary state agents’
constant search must compensate volatility, as articulated by the so-called Red Queen Principle (26, 27): ‘‘...it takes all of the running you can do, to keep in the same place’’ (see more on this below).

The three rates (\( \lambda \), \( \eta \), and \( \xi \)) are the parameters of our model, but one of them can be eliminated by an appropriate time rescaling. We are interested in the properties of the network \( g(t) \) in the stationary state as \( t \to \infty \). Relevant magnitudes in this respect are the density of the network and its clustering. Network density at any time \( n \) is measured by the average node degree \( z(t) \), where the degree \( z_i(t) \) of a node \( i \) is defined by the number of neighbors it has. On the other hand, network clustering \( C(t) \) is obtained by averaging the clustering coefficient \( C_i(t) \) of all nodes \( i \), which is the fraction of pairs of neighbors of \( i \) who are also neighbors among themselves. Although random networks have \( C_i \sim 1/n \), social networks typically have a clustering coefficient (5) bounded above zero.

We carried out extensive numerical simulations by using a discrete time Markov chain to approximate the continuous time process. The results were found to depend very weakly on the specific discretization scheme used (see Fig. 1). For \( \xi = 0 \), the dynamics is very simple and the stationary network is a random graph with average degree \( z = 2\eta/\lambda \) (see below). For \( \eta \ll \lambda \) the network is composed of many disconnected parts. Fig. 1 shows what happens in a computer experiment where the local search rate \( \xi \) is first increased and then decreased very slowly (see Fig. 1 legend). For small \( \xi \), network growth is limited by the global search process that proceeds at rate \( \eta \). Clusters of more than two nodes are rare, and when they form local search quickly saturates the possibilities of forming new links. Suddenly, at a critical value \( \xi_c \), a giant component connecting a finite fraction of the nodes emerges. The average degree \( z \) indeed jumps abruptly at \( \xi_c \). The network becomes more and more densely connected as \( \xi \) increases further. But when \( \xi \) decreases, we observe that the giant component remains stable also beyond the transition point (\( \xi < \xi_c \)). Only at a second point \( \xi_2 \) does the network lose stability and the population gets back to an unconnected state. There is a whole interval \( [\xi_1, \xi_2] \) where both a dense-network phase and one with a nearly empty network coexist. The coexistence region \( [\xi_1, \xi_2] \) shrinks as \( \eta \) increases and it disappears for \( \eta > 0.03\lambda \). This behavior attains already for moderately small \( n \), even though in this case finite size effects are strong; these effects essentially vanish when \( n \approx 1000 \) (see Fig. 1). The average clustering coefficient \( C \) shows a nontrivial behavior. In the unconnected phase, \( C \) increases with \( \xi \) as expected. In this phase, \( C \) is close to one because the expansion of the network is mostly carried out through global search, and local search quickly saturates all possibilities of new connections. On the other hand, in the dense-network phase, \( C \) takes relatively small values. This makes local search very effective. Remarkably, we find that \( C \) decreases with \( \xi \) in this phase, which is rather counterintuitive: by increasing the rate \( \xi \) at which bonds between neighbors form through local search, the density \( C \) of these bonds decreases.

The stability of the dense network phase in the coexistence region confers resilience to the system. It implies that a dense network is robust with respect to deteriorating conditions (higher \( \lambda \) or smaller \( \xi \)) and it may resist even under conditions in which a stable dense network would not form. In fact, similar behavior is found, fixing \( \xi \) and \( \eta \), as a function of the volatility rate \( \lambda \).

The system behavior observed in Fig. 1 is typical of first-order phase transitions and is remarkably similar to the rise of hysteresis in physics, a phenomenon that has its origins in the ergodicity breakdown. Even if, in principle, the process is ergodic, because all configurations can be reached from any other configuration, when \( n \) is large the configuration space gets broken into different ergodic components. Transitions across the boundaries of these components require large deviations that occur only with a probability that is exponentially small in \( n \) (they require fluctuations out of equilibrium in a collection of local neighborhoods whose number is of order \( n \); see below). Strictly speaking, transitions between the two components will occur, but one typically has to wait astronomically large times. The occurrence of phase coexistence in our model is also intuitive and has many analogies with that of a real fluid: the local process \( \xi \) mimics short-range attractive interaction, whereas the \( \lambda \) and \( \eta \) processes capture the effects of temperature and random collisions. Increasing \( \xi \) is analogous to compressing the fluid (reducing the volume), which increases the chances that two molecules enter into the range of mutual interaction. An important difference is that interaction is long ranged in our model, which, as discussed in ref. 28, makes it impossible to have bubbles of one phase into the other: the system is either all in one or in the other phase.

To shed light on these numerical results, we study the dynamics of the distribution \( P(\xi, \hat{C}, t) \) of the degrees \( \xi \) and clustering coefficients \( \hat{C} \). Specifically, we study a mean field approximation that assumes \( \hat{C}_i = C \) for all \( i = 1, \ldots, n \) and

\[
P(\xi, \hat{C}, t) = \prod_{i=1}^{n} p(z_i, t) \delta(C_i - C).
\]

It is convenient to set \( \lambda = 1 \), by an appropriate time rescaling. Then, the transition rates that enter into the master equation for \( p \) are:

\[
w(z_i \to z_i + 1) = 2\eta + \beta(z_i) + \gamma z_i
\]

\[
w(z_i \to z_i - 1) = z_i
\]
where \( \theta(x) = 0 \) for \( x \leq 0 \) and \( \theta(x) = 1 \) for \( x > 0 \). In Eq. 2, the first term accounts for long-distance search (the factor 2 arises because site \( i \) can be either the origin or the destination of the process). The second term arises from local search, and it requires that \( z_i > 1 \). Here \( \beta = \xi(1 - C)P(z_i > 1 / ij \in g) \) is proportional to the conditional probability that \( z_i > 1 \) given that \( ij \in g \). Finally, the last term accounts for indirect local search opportunities given to a friend \( k \) of a friend \( j \) of \( i \). This process is proportional to \( z_i \) and \( \gamma = \xi(1 - C(z_i)^2) \) accounts for the probability that \( i \) is not a friend of \( k \) and that \( k \) selects \( j \).

Both \( \beta \) and \( \gamma \) will be determined self-consistently. Using the generating function method, the master equation for \( p(z) \) in the stationary state can be solved with the result

\[
\pi(s) = \sum_z s^z p(z) = \frac{\beta + 2\eta(1 - \gamma)s^{-\mu}}{\beta + 2\eta(1 - \gamma)s^{-\nu}}, \quad [4]
\]

where \( \mu = (2\eta + \beta)/\gamma \). Simple algebra shows that

\[
p(z) = \frac{1}{\beta + 2\eta(1 - \gamma)s^{-\nu}} \left[ \beta s^{\nu} \left( z / \mu \right)^{\nu} + \frac{2\eta \Gamma(z + 1)}{\Gamma(1)} \right]. \quad [5]
\]

where \( p_0 \) and \( p_1 \) are constants. Notice that \( p(z) \sim z^\mu \) has a power law behavior for small \( z \) and decays exponentially \( p(z) \sim e^{-\delta z^\gamma} \) for large \( z \), in perfect agreement with the behavior observed in numerical simulations.

Eq. 4 allows us to compute the distribution \( P(z_i = u \mid ij \in g) = p(u) \) for the degree \( z_i \) of a neighbor \( j \) of \( i \). The larger \( z_i \) the more likely \( i \) is a neighbor of \( j \). Thus, \( \rho(u) = \mu p(u) \), which, in terms of the generating function implies \( \pi(s) = s^\nu \pi'(1) \). This makes it possible to compute \( P(z_i > 1 / ij \in g) = 1 - \pi'(0) = 1 - \pi'(0)/\pi(1) \) and the factor \( \left( z / \mu \right)^{\nu} = \left[ 1 - \pi(0)/\pi(1) \right] \), which enters the definition of \( \gamma \), thus leaving us with the self-consistent equations:

\[
\beta = \xi(1 - C) \left[ 1 - \frac{\pi'(0)}{\pi(1)} \right] \quad [6]
\]

\[
\gamma = \xi(1 - C) \left[ 1 - \frac{\pi(0)}{\pi(1)} \right]. \quad [7]
\]

As we should, in the limit \( \xi \to 0 \) we find \( \beta, \gamma \to 0 \), and we recover a pure Poisson distribution with mean \( 2\eta \). Under \( \xi > 0 \), local search makes the degree distribution loose its Poisson character. But with constant \( C \), Eqs. 6 and 7 are not able to reproduce the observed behavior. It just predicts a smooth crossover and no phase coexistence. This finding means that, to shed light on the observed behavior, it is essential to allow for \( C \) to depend on the parameters of the model.

To derive an equation for \( C \), we focus on a particular site \( i \) and analyze the process that governs the number \( Q_i = C z_i (z_i - 1)/2 \) of pairs of neighbors of \( i \), which are also neighbors among themselves. Local search contributes to an increase in \( Q_i \) in two ways. The first is when a local search opportunity is given to site \( i \) itself, which has already been discussed above. Its rate is \( W_i(Q_i \to Q_i + 1) = \beta \). The second occurs when a local search opportunity is given to some friend \( j \) of \( i \), who then asks about his other friends \( k \) (\( k \neq j \)). This may lead to the formation of the link between \( j \) and \( k \), thus increasing \( Q_i \) by one. The rate of this process is given by \( W_j(Q_i \to Q_i + 1) = \xi(z_i (z_i - 1))/2(1 - C) \). Here, \( 1/z_i \) is the probability that \( j \) picks \( i \) from his neighbors and \( 1 - C \) is the probability that \( k \) is not a friend of \( i \). This rate should be multiplied by the number \( z_i \) of neighbors of \( j \), but is zero unless \( z_i > 2 \). Finally, we must account for the link-decay process that, contrary to the former two, decreases \( Q_i \). The rate at which this happens is \( W_d(Q_i \to Q_i - 1) = (z_i (z_i - 1)/2)C/2 \).
close to the diluted network solution when $\xi$ is close to $\xi_2$. This finding suggests that, in this region, transitions from diluted to dense networks are more probable than transitions in the other direction, as also observed in numerical simulations.

The behavior of the system as a function of $\Lambda$ (with $\xi$ fixed) can also be read from the phase diagram. It can be traced, as $\Lambda$ increases, by moving toward the origin on a line with constant slope $\eta/\xi$. The coexistence region culminates in a second-order phase transition point at the critical point $\eta/\Lambda \approx 0.226$. Beyond this point, the discontinuous transition turns into a smooth crossover. The specific critical values of the parameters are inaccurate, as is often the case in mean field calculations. The corresponding shift to a social network with low clustering. The situation can be regarded as akin to situations coexist and are metastable, which suggests that external intervention (in the form of a nucleation event) may be needed to trigger the growth of the new phase. The model also underscores the importance of effective search to offset environmental volatility in a highly connected network. This, as explained above, is in line with what is known as the Red Queen Principle in evolutionary biology (27). In our network setup, the effectiveness of search is directly associated to low clustering. This explains that the abrupt transition to the high-dense phase must be mirrored (otherwise it would not be stable) by a corresponding shift to a social network with low clustering.

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