Betti numbers of holomorphic symplectic quotients via arithmetic Fourier transform

Tamás Hausel

Mathematical Institute, University of Oxford, Oxford OX1 3LB, United Kingdom; and Department of Mathematics, University of Texas, Austin, TX 78712

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A Fourier transform technique is introduced for counting the number of solutions of holomorphic moment map equations over a finite field. This technique in turn gives information on Betti numbers of holomorphic symplectic quotients. As a consequence, simple unified proofs are obtained for formulas of Poincaré polynomials of toric hyperkähler varieties (recovering results of Bielawski–Dancer and Hausel–Sturmfels), Poincaré polynomials of Hilbert schemes of points and twisted Atiyah–Drinfeld–Hitchin–Manin (ADHM) spaces of instantons on $\mathbb{C}^2$ (recovering results of Nakajima–Yoshioka), and Poincaré polynomials of all Nakajima quiver varieties. As an application, a proof of a conjecture of Kac on the number of absolutely indecomposable representations of a quiver is announced.

Proposition 1. The number of solutions of the equation $\mu(v, w) = \xi$ over the finite field $\mathbb{F}_q$ equals

$$
\#\{(v, w) \in \mathbb{M}||\mu(v, w) = \xi\} = |g|^{-1/2} |\mathcal{T}(a_\delta) (\xi)|
$$

$$
= |g|^{-1} |\mathcal{V}| \sum_{X \in \Theta} a_\delta(X) \Psi(\langle X, \xi \rangle).
$$

To explain the last two terms in the proposition above, we need to define Fourier transforms (2) of functions $f: g \to \mathbb{C}$ on the finite Lie algebra $g$, which here we think of as an abelian group with its additive structure. To define this fix $\Psi: \mathbb{F}_q \to \mathbb{C}^*$ a nontrivial additive character, and then we define the Fourier transform $\mathcal{T}(f): g^* \to \mathbb{C}$ at a $Y \in g^*$

$$
\mathcal{T}(f)(Y) = |g|^{-1/2} \sum_{X \in \Theta} f(X) \Psi(\langle X, Y \rangle).
$$

Proof: Using two basic properties of Fourier transform

$$
\mathcal{T}(\mathcal{T}(f))(X) = f(-X),
$$

for $X \in g$ and

$$
\sum_{w \in \mathbb{V}^*} \Psi((v, w)) = |V| \delta_0(v),
$$

for $v \in \mathbb{V}$ we get

$$
\#\{(v, w) \in \mathbb{M}||\mu(v, w) = \xi\} = \sum_{\nu \in \mathbb{V}} \sum_{w \in \mathbb{V}^*} \delta_\xi(\mu(v, w))
$$

$$
= \sum_{\nu \in \mathbb{V}} \sum_{w \in \mathbb{V}^*} |g|^{-1/2} \mathcal{T}(\delta_\xi)(-\mu(v, w))
$$

$$
= \sum_{\nu \in \mathbb{V}} \sum_{w \in \mathbb{V}^*} \sum_{X \in \Theta} |g|^{-1/2} \mathcal{T}(\delta_\xi)(X) \Psi(\langle X, -\mu(v, w) \rangle)
$$

$$
= \sum_{\nu \in \mathbb{V}} \sum_{w \in \mathbb{V}^*} \sum_{X \in \Theta} \Psi(-g(X) v)
$$

$$
= \sum_{\nu \in \mathbb{V}} \sum_{w \in \mathbb{V}^*} \sum_{X \in \Theta} |g|^{-1/2} \mathcal{T}(\delta_\xi)(X) |V| \delta_\xi(g(X) v)
$$

$$
= \sum_{\nu \in \mathbb{V}} \sum_{w \in \mathbb{V}^*} \sum_{X \in \Theta} |g|^{-1/2} \mathcal{T}(\delta_\xi)(X) |V| a_\delta(X)
$$

$$
= \sum_{\nu \in \mathbb{V}} \sum_{w \in \mathbb{V}^*} \sum_{X \in \Theta} |g|^{-1} |V| a_\delta(X) \sum_{Y \in g^*} \delta_\xi(Y) \Psi(\langle X, Y \rangle).
$$

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1E-mail: hauser@maths.ox.ac.uk.

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Subspaces of the Poincaré polynomial we need to take the opposite of the count of hyperplanes that contain \( a_0(X) \). This way we recover a result of refs. 4 and 5; for a more recent arithmetic proof see (6):

**Corollary 1.** The Poincaré polynomial of the toric hyperkähler variety is given by

\[
P_s(M(\xi, A)) = h(M_0)(t^2),
\]

where \( B \) is a Gale dual vector configuration of \( A \).

**Hilbert Scheme of \( n \)-Points on \( \mathbb{C}^2 \) and Atiyah–Drinfeld–Hitchin–Manin (ADHM) Spaces**

Here \( G = \text{GL}(V) \), where \( V \) is an \( n \)-dimensional \( \mathbb{C} \) vector space. We need three types of basic representations of \( G \). The adjoint representation \( \rho_{ad} : \text{GL}(V) \to \text{GL}(\text{gl}(V)) \), the defining representation \( \rho_{def} : Id : G \to \text{GL}(V) \), and the trivial representations \( \rho_{triv}^k = 1 : G \to \text{GL}(k^k) \). Fix \( k \) and \( n \). Define \( \mathcal{V} = \text{gl}(V) \otimes V \otimes k^k \), \( \mathcal{M} = \mathcal{V} \otimes \mathcal{V}^* \) and \( \rho : G \to \text{GL}(\mathcal{V}) \) by \( \rho = \rho_{ad} \otimes \rho_{def} \otimes \rho_{triv}^k \).

Then we take the central element \( \xi = \text{Id}_V \in \text{gl}(V) \) and define the twisted ADHM space as

\[
\mathcal{M}(n, k) = \mathcal{M}^{n/k} = \mu^{-1}(\xi)/G,
\]

where \( \mu(A, B, I, J) = [A, B] + IJ \),

with \( A, B \in \text{gl}(V) \), \( I \in \text{Hom}(k^k, \mathcal{V}) \), and \( J \in \text{Hom}(\mathcal{V}, k^k) \).

The space \( \mathcal{M}(n, k) \) is empty when \( k = 0 \) (the trace of a commutator is always zero), diffeomorphic with the Hilbert scheme of \( n \)-points on \( \mathbb{C}^2 \), when \( k = 1 \), and is the twisted version of the ADHM space (7) of \( U(k) \) Yang–Mills instantons of charge \( n \) on \( \mathbb{R}^4 \) (cf. ref. 8). By our main Proposition 1 the number of solutions over \( \mathbb{C} \) is \( \xi \) of the equation

\[
[A, B] + IJ = \text{Id}_V,
\]

is the Fourier transform on \( \mathcal{g} \) of the function \( a_0(X) = |\text{ker}(g(X))| \). First we determine \( a_0(X) \) for \( X \in \mathcal{g} \). By the definition of \( \mathcal{g} \) we have

\[
\text{ker}(g(X)) = \text{ker}(\text{ad}(X)) \times \text{ker}(\text{def}) \otimes k^k,
\]

where \( \text{ad}(X) = [X, \text{ad}(X)] \) and \( \text{def} = [\text{def}, \text{def}] \), then we have

\[
a_0(X) = a_{\text{ad}}(X) a_{\text{def}}(X).
\]

Now Proposition 1 gives us

\[
\#(\mathcal{M}(n, k)) = \frac{1}{|G|} \#\{(v, w) \in \mathbb{C}| \mu(v, w) = \xi\}
\]

\[
= \frac{|\mathcal{V}|}{|G|} \sum_{X \in \mathcal{g}} a_0(X) \Psi((X, \xi))
\]

\[
= \frac{|\mathcal{V}|}{|G|} \sum_{X \in \mathcal{g}} a_{\text{ad}}(X) a_{\text{def}}(X) \Psi((X, \xi)).
\]

We will perform the sum adjoint orbit by adjoint orbit. The adjoint orbits of \( \text{gl}(n) \), according to their Jordan normal forms, fall into types, labeled by \( \mathcal{T}(n) \), which stands for the set of all possible Jordan normal forms of elements in \( \text{gl}(n) \). We denote by \( \mathcal{T}_{\text{reg}}(t) \) the types of the regular (i.e., nonsingular) adjoint orbits, while \( \mathcal{T}_{\text{nil}}(s) = \mathcal{T}(s) \) denotes the types of the nilpotent adjoint orbits, which are just given by partitions of \( s \). First we do the \( k = 0 \) case where we know \( \text{a priori} \), that the count should be 0, because the commutator of any two matrix is always trace-free and thus
cannot equal $\xi$ (for almost all $\eta$). Additionally, if we separate the nilpotent regular parts of our adjoint orbits we get

$$0 = \frac{1}{|G|} \sum_{X \in G} a_{\text{red}}(X) \Psi(\langle X, \xi \rangle)$$

$$= \sum_{n=s+\tau} \sum_{\lambda \in \text{Tur}(s)} \left| \mathcal{G}_\lambda \right| \sum_{r \in \text{Tur}(r)} \frac{\mathcal{G}_r}{C_r} \Psi(\langle X, \xi \rangle),$$

where $C_r$ and, respectively, $\mathcal{G}_r$ denotes the centralizer of an element $X_r$ of $\mathfrak{g}$ of type $\tau$ in the adjoint representation of $G$, respectively, $\mathfrak{g}$ on $\mathfrak{g}$.

So if we define the generating series

$$\Phi_{\text{nil}}^0(T) = 1 + \sum_{s=1}^{\infty} \sum_{\lambda \in \text{Tur}(s)} \frac{\mathcal{G}_\lambda}{C_\lambda} T^s,$$

and

$$\Phi_{\text{reg}}(T) = 1 + \sum_{r=1}^{\infty} \sum_{r \in \text{Tur}(r)} \frac{\mathcal{G}_r}{C_r} \Psi(\langle X, \xi \rangle) T^r,$$

then we have

$$\Phi_{\text{nil}}^0(T) \Phi_{\text{reg}}(T) = 1.$$  

However, $\Phi_{\text{nil}}^0$ is easy to calculate (9)

$$\Phi_{\text{nil}}^0(T) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - T^i q^{j-i}},$$

thus we get

$$\Phi_{\text{reg}}(T) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 - T^i q^{j-i}).$$

Now the general case is easy to deal with:

$$\frac{\#(M(n, k))}{q^{nk}} = \frac{1}{|G|} \sum_{X \in G} a_{\text{red}}(X) \Psi(\langle X, \xi \rangle)$$

$$= \sum_{n=s+\tau} \sum_{\lambda \in \text{Tur}(s)} \frac{\mathcal{G}_\lambda}{C_\lambda} \sum_{r \in \text{Tur}(r)} \frac{\mathcal{G}_r}{C_r} \Psi(\langle X, \xi \rangle).$$

Thus, if we define the grand generating function by

$$\Phi^k(T) = 1 + \sum_{n=1}^{\infty} \#(M(n, k)) \frac{T^n}{q^{nk}},$$

and

$$\Phi_{\text{nil}}^k(T) = 1 + \sum_{s=1}^{\infty} \sum_{\lambda \in \text{Tur}(r)} \frac{\mathcal{G}_\lambda}{C_\lambda} |\text{ker}(X_r)|^k T^s,$$

then for the latter we get similarly to the argument for Eq. 4 in ref. 9 that

$$\Phi_{\text{nil}}^k = \Phi_{\text{nil}}^k(T) \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - T^i q^{j+i-1}}.$$

For the grand generating function then we get

$$\Phi_k(T) = \Phi_{\text{nil}}^k(T) \Phi_{\text{reg}}(T) = \prod_{i=1}^{\infty} \prod_{b=1}^{\infty} \frac{1}{1 - T^i q^{b+i-1}}.$$  

Because the mixed Hodge structure is pure, and this count is polynomial, we get the compactly supported Poincaré polynomial. To get the ordinary Poincaré polynomial, we need to replace $q = 1/t^2$ and multiply the $n$th term in Eq. 6 by $t^{2n}$. This way we get Theorem 2.

**Theorem 2.** The generating function of the Poincaré polynomials of the twisted ADHM spaces are given by

$$\sum_{n=0}^{\infty} P_n(M(k, n)) T^n = \prod_{i=1}^{\infty} \prod_{b=1}^{\infty} \frac{1}{1 - t^{2(n+i-1)} q^{b+n+i-1}}.$$  

This result appeared as corollary 3.10 in ref. 10.

**Quiver Varieties of Nakajima**

Here we recall the definition of the affine version of Nakajima’s quiver varieties (11). Let $Q = (V, E)$ be a quiver, i.e., an oriented graph on a finite set $V = \{1, \ldots, n\}$ with $E \subseteq V \times V$ a finite set of oriented (perhaps multiple and loop) edges. To each vertex $i$ of the graph, we associate two finite dimensional $\mathbb{C}$ vector spaces $V_i$ and $W_i$. We call $(v_1, \ldots, v_n, w_1, \ldots, w_n) = (v, w)$ the dimension vector, where $v_i = \dim(V_i)$ and $w_i = \dim(W_i)$. To these data we associate the grand vector space

$$\bigoplus_{i,j \in E} \text{Hom}(V_i, V_j) \oplus \bigoplus_{i \in V} \text{Hom}(V_i, W_i),$$

the group and its Lie algebra

$$\mathfrak{g}_v = \bigoplus_{i \in V} \mathfrak{gl}(V_i),$$

and the natural representation

$$\rho_{v, w} : G_v \rightarrow \text{GL}(V_{v, w}),$$

with derivative

$$\mathcal{L}_{v, w} : \mathfrak{g}_v \rightarrow \mathfrak{gl}(V_{v, w}).$$

The action is from both left and right on the first term and from the left on the second.

We now have $G_v$ acting on $\mathcal{M}(v, w) = \bigoplus_{v, w} \mathfrak{g}_v \times \mathfrak{g}_v$ preserving the symplectic form with moment map $\mu_{v, w} : \mathfrak{g}_v \times \mathfrak{g}_w \rightarrow \mathfrak{g}_v$ given by Eq. 1. We take now $\xi\mathcal{v} = (Id_{V_1}, \ldots, Id_{V_n}) \in (\mathfrak{g}_v)^{G_v}$, and define the affine Nakajima quiver variety (11) as

$$\mathcal{M}(v, w) = \mu_{v, w}^{-1}(\xi) / G_v.$$  

Here we determine the Betti numbers of $\mathcal{M}(v, w)$ using our main Proposition 1, by calculating the Fourier transform of the function $a_{v, w}$ given in Eq. 2.

First, we introduce, for a dimension vector $w \in \mathbb{N}^n$, the generating function
As in the previous section, our main conjecture 1 of Kac (18). Consequently, formulae arising in the representation theory of quantum groups. When the quiver is star-shaped recent work (T.H., unpublished work and T.H., E. Letellier, and F. Rodriguez–Villegas, unpublished work) calculates these Poincaré polynomials using the character theory of reductive Lie algebras over finite fields (2) and arrives at formulas determined by the Hall–Littlewood symmetric functions (12), which arose as the pure part of Macdonald symmetric polynomials (12). In the case the quiver has no loops, Nakajima (16) gives a combinatorial algorithm for all Betti numbers of quiver varieties, motivated by the representation theory of quantum loop algebras. Finally, through the paper (17) of Crawley-Boevey and Van den Bergh, Poincaré polynomials of quiver varieties are related to the number of absolutely indecomposable representations of quivers in the work of Kac (18), which were eventually completely determined by Hua (19).

In particular, formula 9, when combined with results in refs. 19 and 11 and the Weyl–Kac character formula in the representation theory of Kac–Moody algebras (20), yields a simple proof of conjecture 1 of Kac (18). Consequently, formula 9 can be viewed as a q-deformation of the Weyl–Kac character formula (20).

A detailed study of the above generating function 9, its relationship to the wide variety of examples mentioned above, and details of the proofs of the results of this work will appear elsewhere.