Interference between independent fluctuating condensates

Anatoli Polkovnikov*‡§, Ehud Altman*¶, and Eugene Demler*

*Department of Physics, Harvard University, Cambridge, MA 02138; ‡Department of Physics, Boston University, Boston, MA 02215; and ¶Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

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We consider a problem of interference between two independent condensates that lack true long-range order. We show that their interference pattern contains information about correlation functions within each condensate. As an example, we analyze the interference between a pair of one-dimensional interacting Bose liquids. We find universal scaling of the average fringe contrast with system size and temperature that depends only on the Luttinger parameter. Moreover, the full distribution of the fringe contrast, which is also equivalent to the full counting statistics of the interfering atoms, changes with interaction strength and lends information on high-order correlation functions. We also demonstrate that the interference between two-dimensional condensates at finite temperature can be used as a direct probe of the Kosterlitz–Thouless transition. Finally, we discuss the generalization of our results to describe the interference of a periodic array of independent fluctuating condensates.

An important property of Bose–Einstein condensates is the existence of a coherent macroscopic phase. Thus, a crucial benchmark in the study of such systems was the observation of interference fringes when two independent condensates were allowed to expand and overlap (1). This “two-slit” experiment was carried out with cold atoms in three-dimensional harmonic traps, where a true condensate exists. The interference fringe amplitude should then be proportional to the condensate fraction, as was indeed observed. However, with current trapping technology it is possible to confine the bosonic atoms to one (2–4) or two dimensions (5), where a true condensate may not exist. Instead, these systems are characterized by off-diagonal correlations that either decay as a power-law or decay exponentially in space. What is the interference pattern that arises when two such imperfect condensates are allowed to expand and overlap? This question is not just of general academic interest. Recently there have been a number of experiments showing the interference between independent condensates (see, for example, refs. 6–8).

Here we address this problem theoretically and show that the result depends crucially on the correlations within each condensate. Therefore, such an experiment would provide a direct and simple probe of the spatial phase correlations. In principle, spatial phase correlations may also be extracted from juggling experiments (9–11) or the momentum distribution measured by the free expansion of a single condensate (2). However, creating strongly interacting low-dimensional systems typically requires using low-density atomic gases, which makes juggling experiments very challenging. In addition, the highly anisotropic expansion of low-dimensional condensates inhibit measurements of the momentum distribution in the slowly expanding longitudinal direction. A method for probing the phase correlations directly in real space would therefore be very useful.

Results

The simplest geometry that we consider is illustrated in Fig. 1 and consists of two parallel one-dimensional condensates a distance, $d$, apart. After the atoms are released from the trap, they are allowed to expand to a transverse size much larger than $d$, although no significant expansion occurs in the axial direction. An absorption image is then taken by a probe beam directed along the condensate axis. A similar setup is considered for two-dimensional condensates on parallel planes (see Fig. 2). As usual, the absorption image gives the instantaneous three-dimensional density profile integrated along the beam axis, $\rho(x) = \int_0^L dz \, \hat{a}_{\text{tot}}^\dagger(x, z) \, \hat{a}_{\text{tot}}(x, z)$, where $\hat{a}_{\text{tot}}^\dagger$ are the Bose creation operators with the subscript “time of flight” (tof), emphasizing that the corresponding operators are taken after free expansion of atoms, $z$ is the axial coordinate and $x$ is the coordinate along the detector (see Fig. 1). The length $L$ is typically given by the focal length of the imaging beam. It may also be controlled more precisely by applying magnetic field gradients so that only a specified section of the cloud is resonant with the probe light. In principle, one can consider an experiment with a probe beam orthogonal to the plane containing two parallel one-dimensional condensates. In this case, it is possible to integrate the resulting interference image within an arbitrary interval and obtain dependence of the interference contrast on $L$ (note that this dependence characterizes a single run and a series of experiments is still needed to find the average contrast). The other advantage of this setup is that it can reveal the presence of dipolar oscillations in individual condensates. These modes correspond to the center of mass motion and are not affected by interactions. Dipolar oscillations induce an overall tilt in the interference peak position and can be easily removed by integrating $\rho(x)$ along a line tilted with respect to the $z$ axis. However,

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Abbreviation: KT, Kosterlitz–Thouless.

§To whom correspondence should be addressed. E-mail: asp@bu.edu.

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because most of the current experimental systems do not allow imaging beams that are perpendicular to one-dimensional condensates, we concentrate on the setup shown in Fig. 1.

To discuss the interference contrast, we consider the correlation function of the density operator,

\[ \langle p(x_1)p(x_2) \rangle = \delta(x_1 - x_2) \int_0^L dz \rho(z) + \int_0^L dz dz' a_1^\dagger(z_1) a_2^\dagger(z_2) a_{1\omega}(z_1) a_{2\omega}(z_2), \]

where \( r_i \) stands for \((x_i, z_i)\). Single-particle operators in Eq. 1 should be taken after the expansion time \( t \). We can relate them to operators before the expansion (12): \( a_{\omega\omega}(x, z) = a_{1\omega}(z) e^{iQ_{x\omega}z - iQ_{\omega\omega}z^2/2m} + a_{2\omega}(z) e^{iQ_{x\omega}z - iQ_{\omega\omega}z^2/2m} \), with \( a_1 \) and \( a_2 \) being operators in the two condensates and \( Q_1 \) and \( Q_2 = m(x \pm d/2)/\hbar t \).

We therefore find that the correlation function in Eq. 1 has an oscillating component at wave vector \( Q \) with a nontrivial power of the imaging length. In the noninteracting limit \( \xi \) is the healing length, which also depends only on \( L/\xi \), Eq. 4 can be generalized for the case of parallel two-dimensional condensates by taking \( z \) to represent the planar coordinates.

To gain intuition into the physical meaning of the fringe contrast, we consider the situation where \( (a_1^\dagger(z)a_2(0)) \) decays exponentially with distance with a correlation length \( \xi \ll L \). Then, Eq. 4 implies that \( |A_Q| \propto \sqrt{L/\xi} \), which has a very simple physical meaning. Because the phase is only coherent over a length, \( \xi \), the system is effectively equivalent to parallel chains with \( L/\xi \) pairs of independent condensates.

If the two condensates are identical (but still independent), we may simplify Eq. 3:

\[ \langle |A_Q|^2 \rangle = L \int_0^L dz \langle a_1^\dagger(z)a_1(0) \rangle^2. \]

Here, we neglected boundary effects by integrating over the center of mass coordinate and assuming that the correlations depend only on \( (z_1 - z_2) \). Eq. 4 can be generalized for the case of parallel two-dimensional condensates by taking \( z \) to represent the planar coordinates.

To determine the amplitude of the interference fringes in individual measurements, one should consider an expectation value of the quantity that does not involve the random phase of \( A_Q \). This consideration naturally brings us to Eq. 3. From shot to shot, \( |A_Q|^2 \) fluctuates as well, and Eq. 3 gives its average value.

One-Dimensional Bose Liquids. We proceed to discuss the case of a one-dimensional interacting gas. We first consider a system at sufficiently low temperature, \( \xi_1 \gg L \), where \( \xi_1 \) is the temperature-dependent correlation length defined in Eqs. 17 and 18. In this regime, the correlations decay as a power-law rather than decay exponentially. We therefore, expect that the fringe amplitude will somehow interpolate between the two simple limits considered above. Specifically, at long wavelengths, the one-dimensional Bose gas is described by a Luttinger liquid (16), and the long-distance off-diagonal correlations behave as

\[ \langle a_1^\dagger(z)a_2(0) \rangle \sim \rho \left( \frac{\xi_0}{z} \right)^{1/2K}. \]

Here, \( \rho \) is the particle density, \( \xi_0 \) is the healing length, which also serves as the short-range cutoff, and \( K \) is the Luttinger parameter. For bosons with a repulsive short-range potential, \( K \) ranges between \( 1 \) and \( \infty \), with \( K = 1 \) corresponding to strong interactions, or “impenetrable” bosons, and \( K \to \infty \) corresponding to noninteracting bosons. Substituting Eq. 5 into Eq. 4 and assuming that \( L \gg \xi_0 \), we arrive at one of our main results:

\[ \langle |A_Q|^2 \rangle = C \rho^2 L^3 \left( \frac{\xi_0}{L} \right)^{1/K}, \]

where \( C \) is a constant of order unity. Thus, we see that the amplitude of the interference fringes \( A_Q = \sqrt{\langle |A_Q|^2 \rangle} \) scales with a nontrivial power of the imaging length. In the noninteracting limit \( K \to \infty \) the scaling is linear \( A_Q \propto L \) as expected for a fully coherent system. Interestingly \( A_Q \propto \sqrt{L} \) in the hard-core
should consider higher moments of the operator and experimental run to the next. To answer this question, one interesting question is how this amplitude fluctuates from one corrections to the power-law scaling.

Gordon problem. This connection gives a direct link between the scaling of the fringe amplitude with system size or by Ref. 18 discusses the details of this problem and the crossover interference of two one-dimensional quasicondensates and ther-modynamic analysis.

We now consider a pair of parallel two-dimensional Bose systems. In direct analogy to the one-dimensional condensates, the imaging axis may be taken parallel or at some angle to the plane of the condensates. In the former case, one should consider the scaling of the fringe amplitude with imaging length, whereas, in the latter case, one should consider the variation with angle. It is well known that in two dimensions long-range order may exist only at zero temperature. At sufficiently low temperatures, off-diagonal correlations are algebraic, with

\[ \langle a^\dagger(r)a(0) \rangle \sim \rho \left( \frac{\xi_a}{r} \right)^\alpha, \tag{10} \]

for \( r \gg \xi_a \). In contrast, above the Kosterlitz–Thouless (KT) transition at \( T = T_c \), the correlations decay exponentially. We will show that this transition is characterized by a jump in the behavior of the fringe amplitude, related to the well known universal jump of the superfluid stiffness at \( T_c \).

The exponent in Eq. 10 is given by \( \alpha = mT/2\pi\hbar(T)h^2 \). For weakly interacting bosons at temperatures well below \( T_c \), \( \rho(T) \) is simply equal to the density, \( \rho \). As one approaches the transition, \( \rho \) is renormalized by fluctuations, and, at the transition, \( \rho(T_c) = 2mT_c/\pi\hbar^2 \). Therefore, the exponent \( \alpha \) assumes a universal value \( \alpha_c = 1/4 \) at the transition. Thus, for temperatures \( T < T_c \), we have \( 0 < \alpha < \alpha_c \).

Let us now discuss the consequences of this physics to the experimentally measurable fringe amplitude. As illustrated in Fig. 1, the interference pattern is now truly two-dimensional in the sense that cuts along \( x \) at different coordinate, \( y \), display a different fringe pattern. To obtain a one-dimensional pattern as a function of \( x \) alone, we may integrate the image intensity over an “integration length,” \( L_z \). Recall that, in addition, the imaging process automatically integrates over an imaging length \( L_z \) along the \( z \) axis. Now the generalization of Eq. 4 to the two-dimensional case is straightforward:

\[ L \int_0^L dq(z) \cos[q(\theta)z]. \tag{7} \]
\[ \langle |A_Q|^2 \rangle \sim L_y L_z \int_0^{L_y} dy \int_0^{L_z} dz a^\dagger(y, z) a(0, 0)^2. \]  

[11]

For simplicity, we assume that \( L_y \) and \( L_z \) are scaled simultaneously as \( L_y = L_z = \sqrt{\Omega} \), with \( \Omega \) being the imaging area. Then, with Eqs. 10 and 11, we find that, for \( T < T_c \),

\[ \langle |A_Q|^2 \rangle \sim \Omega^{2 - \alpha}. \]  

[12]

So, below the transition, the scaling of \( |A_Q| \) with size ranges from linear at \( T = 0 \to \Omega^{0.875} \) at the KT transition (\( \alpha_c = 1/4 \)). In contrast, for \( T > T_c \), the correlations decay exponentially and \( |A_Q| \sim \sqrt{\Omega} \). Hence, we find a universal jump at \( T = T_c \) in the power characterizing the size dependence of \( |A_Q| \). One can also consider a setup for which only dependence on one length, say \( L_y \), is studied, whereas the other one, \( L_z \), is fixed. Then, if \( L_y \gg L_z \), Eq. 12 immediately generalizes to \( \langle |A_Q|^2 \rangle \sim L_z^{-2 \alpha - \epsilon} \). In this case, the power jump in \( |A_Q| \) is bigger: from 0.75 to 0.5 as \( T \) crosses \( T_c \).

This jump is a direct signature of the KT physics to be contrasted with the result for one-dimensional condensates, for which the scaling power with system size interpolated smoothly between 1 and 2. It should be noted, however, that a one-dimensional system on an optical lattice, which undergoes a Mott transition at \( T = 0 \), would display a universal jump similar to the two-dimensional case discussed here. In the same way, one can study the shape of the distribution of the interference amplitude and find that, as \( T \) increases to \( T_c \), the distribution gradually broadens but always remains relatively narrow. In contrast, as \( T \) becomes larger than \( T_c \), the distribution assumes a broad Poissonian form.

The analysis for imaging the two-dimensional condensates with a slanted probe beam can be carried over from the one-dimensional case. The scaling of the interference contrast with \( q = k_0 \tan \theta \), at constant imaging area, is then \( \langle |A_Q|^2 \rangle \sim 1/q^2 \) below the KT transition, and \( \langle |A_Q|^2 \rangle \sim 1/(1 + q^2 \xi^2)^{1/2} \) above it. Again the transition is characterized by a universal jump of the power at small \( q \). We emphasize that \( \theta \) can be either the angle between the beam and the \( z \) axis (see Fig. 1) or the angle between the \( y \) axis and the direction of integration. The latter is preferable, because, within a single experimental shot, it is possible to obtain the whole angular dependence of \( A_Q^2 \).

Regardless of the experimental approach of choice, the interference between a parallel pair of independent two-dimensional condensates can serve as a direct probe of KT physics. However, a word of caution is in order. The correlation length, which coincides with the healing length at very low temperatures (19) \( (\xi_T = \xi_h = \hbar/\sqrt{m g n} \) diverges at the KT transition as \( \xi_T \sim \exp(b/\sqrt{T - T_c}) \) (20). Therefore, with increasing temperature, one has to probe the system at increasing distances \( r \gg \xi_h(T) \) [or \( L^{-1} \ll q \ll 1/\xi_h(T) \)] to measure the asymptotic form of the correlation function given by Eq. 10, which might hinder accurate determination of the universal jump.

**Discussion**

We considered a pair of interfering quasicondensates; however, most of our arguments can be generalized to the case of several independent condensates. Of particular interest is a periodic array of tubes (2–4) or pancakes created by an optical potential (5, 21, 22). The interference pattern in this case shows correlations at a set of wave vectors \( Q_n = nQ \), where \( n \) is an integer and \( Q \) is determined by the distance between neighboring condensates. The size and angle dependence of the average interference amplitudes for each of these wave vectors should have the same scaling properties as two quasicondensates. However, the distribution function of fringe amplitudes will be different. In particular, in the limit of a large number of condensates, the distribution function should become very narrow. This result follows immediately from the observation that in this limit, higher-order correlation functions in TOF images are dominated by products of two point correlation function in different condensates, so there should be no broadening associated with Eq. 15 below. Another point worth making regards the possibility of making analogous experiments with Fermions. For example, one can consider an interference of two independent one-dimensional fermionic systems. One obvious difference from the bosonic case will be the change of sign in the correlation function (see Eq. 2), reflecting different statistics of the fermions (this corresponds to fermion antibunching). More importantly, the correlation function decays as \( 1/|x|^{1/2 + k \Delta K r} \), i.e., as \( 1/x \) or faster. This scaling means that the integral in Eq. 4 is dominated by short distances, at which the Luttinger liquid description is not sufficient, and that the integral converges as \( L \to \infty \). Infrared convergence of Eq. 4 implies trivial scaling \( |A_Q| \propto \sqrt{L} \). Moreover, the integrals appearing in all moments of the distribution are similarly infrared convergent, which results in a Poissonian fringe distribution at large \( L \) values: \( P(x) \propto e^{-x} \), as found for bosons at high temperature. To extract information on the Luttinger parameter, one can analyze the decay of density–density correlations in the noise \( \rho_{xx}(x, z_0) \rho_{xx}(x, z_0 + z) \) as a function of \( z \), which is directly related to the integrand of Eq. 4. We note that these correlations have an oscillating component similar to Friedel oscillations, with wave vector \( 2k_f \). The oscillating component appears as a peak (cusp singularity) in the angular dependence of the interference contrast at an imaging angle \( k_0 \tan \theta \approx 2k_f \) (see Eq. 7).

The shape of this cusp as well as of the cusp at \( \theta = 0 \) holds information on the Luttinger parameter. A more-detailed analysis of the fermionic case lies beyond the scope of this work.

In conclusion, we analyzed the interference between two independent quasicondensates. We showed that scaling properties of interference fringes directly probe the algebraic off-diagonal correlations. In particular, for one-dimensional condensates, the scaling with imaging length or with temperature allows the extraction of the Luttinger parameter. In the case of two-dimensional condensates, this method provides a unique probe of the KT transition. We also argued that, in the one-dimensional case, one can use the distribution function of the interference amplitude (which is also equivalent to the full counting statistics of interfering bosons) as the qualitative probe of the Luttinger constant. In particular, at \( K \gg 1 \) the distribution is narrow and at \( K \to 1 \) or, at finite temperatures, it becomes wide Poissonian (see Fig. 2). In the two-dimensional case, we expect a sharp change in the shape of the distribution function at the KT transition. The scaling analysis remains intact if more than two independent condensates are present, but the distribution functions can no longer be used as a probe of the correlations.

**Methods**

**Luttinger Liquid Parameter.** The Luttinger liquid provides a universal long-wavelength description of one-dimensional, interacting Bose liquids that allows the calculation of the long-distance behavior of correlations such as Eq. 5. In certain regimes, it is possible to derive the Luttinger parameter, \( K \), and the healing length, \( \xi_h \), from the microscopic interactions. In particular, for bosons with weak contact interactions, relevant for ultra-cold atom systems discussed in this work, one can use Bogoliubov theory to obtain (23, 24)

\[ \xi_h \approx \frac{\hbar}{\sqrt{\gamma}}, \quad K \approx \frac{\pi}{\sqrt{\gamma}} \left( 1 - \frac{1}{2\pi} \right)^{1/2}. \]  

[13]

Here, \( \gamma = 2/(a_{id} \rho) \leq 1 \) is a dimensionless measure of the interaction strength, and \( a_{id} \) is the one-dimensional scattering
length. Analytic expressions for these parameters may also be derived in the limit of strong contact interaction $\gamma \gg 1$ (23):

$$\xi = 1/\rho, \quad K = 1 + \frac{4}{\gamma}$$

[14]

**Moments of the Fringe Amplitude.** All of the moments in the distribution of $|A_0|^2$ can be obtained by generalizing the two-point correlation function in Eq. 3 to the 2n-point correlation function

$$\langle |A_0|^{2n} \rangle = \int d\omega_2 \ldots \int d\omega_n$$

$$\langle a_1^{\dagger}(z_1) \ldots a_1^{\dagger}(z_n) a(z_i') \ldots a(z_i') \rangle^2.$$  

[15]

For bosons with repulsive interactions described by the Luttinger parameter, $K$, we have

$$\langle |A_0|^{2n} \rangle = C L^{2n} \left(\frac{T}{\hbar^2} \right)^n \prod_{i=1}^n \int d\omega_i \prod_{i<j} \left( \frac{\omega_i - \omega_j}{\omega_i - \omega_j} \right)^{1/K} \left( \frac{\omega_i - \omega_j}{\omega_i - \omega_j} \right)^{1/K}$$

[16]

which is precisely of the form $\langle |A_0|^{2n} \rangle = \langle |A_0|^{2n} \rangle F_n(K)$. Integrals of the type appearing in Eq. 16 have been discussed by Fendley et al. (25), who demonstrated that they can be calculated by using special properties of Jack polynomials. From the knowledge of all moments, one can, in principle, construct the full distribution of the interference fringe amplitude. In this paper, we only discuss the limits of weak ($K \gg 1$) and strong ($K \approx 1$) interactions.

**Finite Temperature Correlations in One Dimension.** The finite-temperature, off-diagonal correlations are given by ref. 23:

$$\langle a^{\dagger}(z)a(0) \rangle \sim \rho_{nh}^{1/2K} \left( \frac{\pi/\xi_T}{\sinh \pi z/\xi_T} \right)^{1/2K},$$

[17]

where the thermal correlation length, $\xi_T$, is

$$\xi_T = \left( \frac{\hbar^2}{m \xi_0^2} \right)^{1/2} T.$$  

[18]

Eq. 17 is valid for sufficiently low temperatures so that $\xi_T \gg \xi_0$ or, equivalently, $T \ll \hbar^2/m \xi_0^2$. For $z \ll \xi_T$, Eq. 17 reduces to the zero temperature correlation (5). In the opposite limit, $z \gg \xi_T$, the correlation function given by Eq. 17 may be approximated by

$$\langle a^{\dagger}(z)a(0) \rangle \sim \rho_{nh}^{1/2K} e^{-z^2/2K_T}.$$  

[19]

As we already noted, for sufficiently low temperatures when $\xi_T \gg L$, the fringe amplitude may be found from Eq. 6. For $L \gg \xi_T$, Eq. 19 implies

$$\langle |A_0|^2 \rangle \sim \rho_{nh}^{1/K} \xi_T^{-1/2} \frac{K}{\pi}.$$  

[20]

Finally, substituting Eq. 18 for $\xi_T$ gives Eq. 9.

We also note that the angular dependence of the fringe amplitude at finite temperature is given by

$$\langle |A_0|^2 \rangle \sim \rho_{nh}^{1/K} \xi_T^{-1/2} \frac{K}{\pi} \left( 1 + K^2 q^2 T^2 / \pi^2 \right).$$  

[21]

From this expression, it is harder to extract $K$ directly because of uncertainty in the determination of $\xi_0$ and, hence, $\xi_T$.

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