Analysis of a Picard modular group

Gábor Francsics*† and Peter D. Lax‡§

*Department of Mathematics, Michigan State University, East Lansing, MI 48824; and ‡Department of Mathematics, Courant Institute of Mathematical Sciences, New York University, New York, NY 10012

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Our main goal is to analyze the geometric and spectral properties of the Picard modular group with Gaussian integer entries acting on the two-dimensional complex hyperbolic space.

complex hyperbolic space | fundamental domain | point spectrum | discrete automorphism group | Maass cusp form

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ur main goal is to analyze the geometric and spectral properties of the Picard modular group \( \Gamma = SU(2, 1; \mathbb{Z}[i]) \) acting on the complex hyperbolic space \( \mathbb{C}H \). The complex hyperbolic space \( \mathbb{C}H \) is the rank one Hermitian symmetric space of noncompact type, \( SU(n, 1)/SU(n) \) \( \times U(1) \). A standard model of the complex hyperbolic space is the complex unit ball \( B^n = \{ z \in \mathbb{C}^n; |z| < 1 \} \) with the Bergman metric \( g = \sum_{k=1}^{n} g_{z_k} dz_k \otimes d\bar{z}_k \) where \( g_{z_k} = \text{const} \log(1 - |z|^2) \). This model is the bounded realization of the Hermitian symmetric space \( \mathbb{C}H \). We shall use mainly the unbounded hyperquadratic model of the complex hyperbolic space, that is \( D^n = \{ z \in \mathbb{C}^n; \exists x_{2k} > \frac{1}{4} \sum_{j=1}^{n} |z_j|^2 \} \).

The Picard modular groups are

\[ SU(n, 1; \mathbb{C}_d), \]

where \( \mathbb{C}_d \) is the ring of algebraic integers of the imaginary quadratic extension \( \mathbb{Q}(\sqrt{d}) \) for any positive squarefree integer \( d \) (see ref. 1). We are interested in the simplest case perhaps: \( d = 1 \), that is, \( \mathbb{C}_1 = \mathbb{Z}[i] \), the Picard modular group with Gaussian integer entries. The Picard modular group \( SU(n, 1; \mathbb{Z}[i]) \) is a discontinuous holomorphic automorphism subgroup of \( \mathbb{C}H \) with Gaussian integer entries. It is a higher-dimensional analogue of the modular group, \( \text{PSL}(2, \mathbb{Z}) \), in \( \mathbb{C}^n \).

Geometric and spectral properties of lattices in symmetric spaces attracted much attention during the last decades. Although remarkable progress has been achieved, several important problems related to arithmeticity, existence of embedded eigenvalues in the continuous spectrum, etc. are still open. The general structure of a fundamental domain for lattices is well known since the work of Garland and Raghunathan (2), for example. However, there are very few fundamental domains known completely explicitly. This statement is especially true for complex hyperbolic spaces. The case of complex hyperbolic spaces is a particularly difficult case. This phenomenon is well known since the work of Mostow (3). Recently, very strong progress has been made in constructing explicit fundamental domains for discrete subgroups of complex hyperbolic spaces; see, for example, the work of Cohn (4), Holzapfel (1, 5), Goldman (6), Goldman and Parker (7), Falbel and Parker (8, 9), Schwartz (10), and Francsics and Lax (11, 12). However, explicit fundamental domains do not seem to be known in the literature for the Picard modular groups, except in the case \( d = 3 \) (see the comment on p. 2 of ref. 9). Moreover, very little is known about the spectral properties of the automorphic complex hyperbolic Laplace–Beltrami operator; see the work of Epstein et al. (13), Reznikov (14), and Lindenstrauss and Venkatesh (15).

The holomorphic automorphism group of \( \mathbb{C}H \), \( \text{Aut}(\mathbb{C}H) \), consists of rational functions \( g = (g_1, \ldots, g_n) : D^n \to D^n \)

\[ g_{z_k}(z) = \frac{a_{j+1k} + \sum_{k=1}^{n} a_{j+1,k}z_k^{j-1}}{a_{1j} + \sum_{k=1}^{n} a_{1,k}z_k^{j-1}}, \]

\( j = 1, \ldots, n \). These automorphisms act linearly in homogeneous coordinates \( \xi_0, \ldots, \xi_n, z_j = \xi_j/\xi_0, j = 1, \ldots, n \). The corresponding matrix \( A = [a_{jk}]_{j,k=1}^{n} \) satisfies the condition

\[ A^{*}CA = C, \]

where

\[ C = \begin{pmatrix} 0 & 0 & i \\ 0 & I_{n-1} & 0 \\ -i & 0 & 0 \end{pmatrix}, \]

and \( I_{n-1} \) is the \( (n-1) \times (n-1) \) identity matrix. The determinant of the matrix \( A \) is normalized to be equal to 1. The matrix \( C \) is the matrix of the quadratic form of the defining function of \( D^n \) written in homogeneous coordinates. Three important classes of holomorphic automorphisms are Heisenberg translations, dilations, and rotations. The Heisenberg translation by \( a \in \mathbb{R}D^2; N_a \in \text{Aut}(\mathbb{C}H) \) is defined as \( N_a(z_1, z_2) = (z_1 + a_1 z_1 + a_2 + i\bar{a})z_2 \). The holomorphic automorphism of \( D^2; A_d(z) = (\delta z_1, \delta^2 z_2) \) is called dilation with parameter \( \delta > 0 \). Rotation in the first variable by \( e^{i\theta}, M_{\theta} = (e^{i\theta}z_1, z_2) \) is a holomorphic automorphism of \( D^2 \) with \( \theta \in \mathbb{R} \). The holomorphic involution

\[ J(z_1, z_2) = (iz_1/z_2, -1/z_2), \]

will also play a significant role.

In a metric space \((X, d)\), the subset \( S \subset X \) is a Siegel set for an isometry group \( G \) of \( X \) if (i) for all \( x \) in \( X \) there is \( g \in G \) such that \( gx \in S \), and (ii) the set \( g(S) \cap S \neq \emptyset \) is finite. Let \( S_{1/4} \) be the set

\[ S_{1/4} = \{ z \in D^2; 0 \leq \text{Re} z_1, 0 \leq \text{Re} z_2, \text{Re} z_1 + \text{Re} z_2 \leq 1, |\text{Re} z_2| \leq \frac{1}{2}, \text{Re} z_2 - \frac{1}{2} |z_2|^2 \geq \frac{1}{4} \}. \]

We introduce horospherical coordinates

\[ (x_1, x_2, x_3, y) \in \mathbb{R}^3 \times \mathbb{R}^+, \]

as \( z_1 = x_1 + ix_2, z_2 = x_3 + i(y + (x_1^2 + x_2^2)/2) \) on \( D^2 \). Then the complex hyperbolic Laplace–Beltrami operator in horospherical coordinates is given by

\[ \Delta_{\mathbb{C}H} = \frac{y}{2}(\partial_x^2 + \partial_y^2) + \frac{y}{2} (2y + x_1^2 + x_2^2) \partial_x^2 + yx_2 \partial_x \partial_y, \]

\[ -yx_1 \partial_x \partial_y - y \partial_y. \]

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To whom correspondence may be addressed. E-mail: francsic@math.msu.edu or lax@cims.nyu.edu.

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We will use the notation

\[ H_f(f) = 2 \int_\mathbb{R} |f|^2 \frac{1}{y^{3/2}} \, dx dy, \]

for the invariant \( L_2 \)-integral, and \( D_f \) for the corresponding Dirichlet integral.

**Statement of the Results**

Our first result is a semiexplicit fundamental domain for the Picard modular group with Gaussian integers.

**Theorem 1.** The set \( S_{1/4} \) is a Siegel set for the Picard modular group \( \Gamma \) in \( \mathbb{C}^2 \). Let \( H_1 = J, H_2, \ldots, H_N \) be all the holomorphic automorphisms \( H \in \Gamma \Gamma_r \) such that \( S_{1/4} \cap H(S_{1/4}) \neq \emptyset \). Then the set

\[ \mathcal{T} = \{ z \in S_{1/4}; |\det H'(z)|^2 \leq 1, \quad 1 \leq j \leq N \} \tag{3} \]

is a fundamental domain of the Picard modular group with Gaussian integers.

At this point the transformations \( H_2, \ldots, H_N \) are not known explicitly; moreover, not all of them are necessary for defining the fundamental domain \( \mathcal{T} \). However, this form of the fundamental domain is already useful for obtaining important geometric and spectral properties.

Three important geometric properties of the fundamental domain \( \mathcal{T} \) are stated in the following theorem.

**Theorem 2.**

(i) The Siegel set \( S_{1/4} \) is invariant under the involutive transformation \( S(z_1, z_2) = (i\bar{z}_1, -\bar{z}_2) \). Moreover, the set of transformations \( H_1, \ldots, H_N \) is invariant under the conjugation

\[ H \mapsto \text{SHS}. \]

Therefore, the fundamental domain \( \mathcal{T} \) is invariant under the involutive transformation

\[ S(z_1, z_2) = (i\bar{z}_1, -\bar{z}_2). \]

(ii) The two-dimensional edge of the fundamental domain \( \mathcal{T} \) at \( z_1 = 0 \) is identical to the standard fundamental domain for the modular group.

(iii) The fundamental domain \( \mathcal{T} \) has a product structure near infinity, that is

\[ \mathcal{T} \cap \{ z \in \mathbb{C}^2; \exists z_2 \approx a \} = S_{1/4} \cap \{ z \in \mathbb{C}^2; \exists z_2 \approx a \}, \]

for large \( a > 0 \).

We exploit the involutive transformation \( S \) to obtain spectral information on the Laplace operator of the Picard modular group \( \Gamma \) in the next theorem. We recall that the continuous spectrum of the Laplace–Beltrami operator \( \Delta_r \) is \( (n^2/4, \infty) \), and an eigenvalue in the continuous spectrum is called embedded eigenvalue or Maass cusp form.

**Theorem 3.** The invariance of the fundamental domain \( \mathcal{T} \) under the transformation \( S \) in Theorem 2 implies the existence of infinitely many embedded eigenvalues in the continuous spectrum of the associated automorphic complex hyperbolic Laplace–Beltrami operator.

Our next goal is to determine a completely explicit fundamental domain for the Picard modular group \( \Gamma \) based on

**Theorem 4.** A fundamental domain for the Picard modular group is

\[ \mathcal{T} = \{ z \in S_{1/4}; |z_1|^2 \geq 1, \quad |r + i - (1 + i)z_1 + z_2|^2 \geq 1 \}
\]

At this point the transformations \( H_2, \ldots, H_N \) are stated in the following theorem.

**Theorem 5.** There are eight holomorphic automorphisms \( G_1 = J, G_2, \ldots, G_8 \) in the Picard modular group \( \Gamma \), described below in Eqs. 4–7, such that the set

\[ \mathcal{T} = \{ z \in S_{1/4}; |z_2|^2 \geq 1, \quad \text{det} G'(z)|^2 \leq 1, \quad j = 2, \ldots, 8 \}, \]

is a fundamental domain of the Picard modular group acting on the complex hyperbolic space \( \mathbb{CH}^2 \). All eight transformations are needed. The holomorphic transformations \( G_1, \ldots, G_8 \) can be described as follows.

There are four transformations with dilation parameter \( 1 \) as follows:

\[ G_1(z_1, z_2) = J(z_1, z_2) = \left( \frac{i\bar{z}_1}{z_2} - \frac{1}{z_2} \right), \tag{4} \]

\[ G_{r+3} = J \circ P_{r+3} = J \circ N_{(1+i \epsilon_j)^{-1}} \circ M_{-1}, \tag{5} \]

with \( r = -1, 0, 1 \).

There are four transformations with dilation parameter \( \sqrt{2} \) as follows:

\[ G_{1/2}^1 = N \left( \frac{1+i}{\sqrt{2}} \right) \circ J \circ P_{1/2} + i \frac{1+i}{\sqrt{2}} \tag{6} \]

and

\[ G_{1/2}^2 = N \left( \frac{1+i}{\sqrt{2}} \right) \circ J \circ P_{1/2} - i \frac{1+i}{\sqrt{2}} \tag{7} \]

where \( r = -1, 1 \).

The precise definition of the holomorphic automorphisms \( P, J, N, A, \) and \( M \) is described in the introduction. We mention that the inequalities in the description of \( \mathcal{T} \) in Theorem 4 are simplified explicit versions of the inequalities.
of Theorem 5.

Outline of the Method

The main building block in our fundamental domain construction is the Siegel set $S_{1/4}$. The triangular shape of $S_{1/4}$ in the $z_1$ variable is the consequence of the fact that a Heisenberg translation $N_0$ is in $\Gamma$ if and only if $\Re a_1, \Re a_1, \Re a_2 \in \mathbb{Z}$ and $|a_1|^2$ is even. The finiteness property of $S_{1/4}$ is obtained by using the transformation formula of the Bergman kernel function and the invariance $J$. We build a semiexplicit fundamental domain $\mathcal{F}$ from the Siegel set $S_{1/4}$ in the following way. Let $H_1 = J, H_2, \ldots,$ $H_N$ be all the holomorphic automorphisms $H \in \Gamma \setminus \Gamma_\ast$ such that $S_{1/4} \cap H(S_{1/4}) = \emptyset$. Then we prove that the set

$$
\{z \in S_{1/4}; |\det H'(z)|^2 \leq 1, 1 \leq j \leq N\},
$$
is a fundamental domain for the Picard modular group acting on the complex hyperbolic space $\mathbf{CH}^2$. At this point, the transformations $H_j, j = 2, \ldots, N$ are not known explicitly; moreover, not all of them contribute to the fundamental domain $\mathcal{F}$.

The key observation in obtaining the spectral properties of $\Delta \Gamma$ is that the transformation $S$ splits the space of $L_2$ automorphic functions into even and odd automorphic functions with respect to the transformation $S$. One can prove that the resolvent of $\Delta \Gamma$ is compact on the space of odd automorphic functions. This step uses a Poincaré inequality in the $x$-variables. Near infinity the fundamental domain has a compact cross section; that is, the cross section written in horospherical coordinates, $K_a = \mathcal{F} \cap \{y = a\}$, is compact for large $a > 0$. The Poincaré inequality is applied on the cross section $K_a$.

The basic idea of the explicit construction in Theorems 4 and 5 can be described easily. Let $\mathcal{F}_1 = S(L) \cap \{z \in \mathbb{C}^2; |z_2| \geq 1\} = S(L) \cap \{z \in \mathbb{C}^2; |\det H'(z)|^2 \leq 1\}$. Clearly, $\mathcal{F} \subset \mathcal{F}_1$. We will prove that if $H$ is one of the transformations $H_2, \ldots, H_N$ in the description of $\mathcal{F}$ in Eq. 3 then either

1. $|\det H'(z)| \leq 1$ for all $z \in \mathcal{F}_1$; or
2. there is a transformation $G_j, j = 2, \ldots, 8$ appearing in Eqs. 5–7 such that

$$
|\det H'(z)| \leq |\det G_j'(z)|,
$$

for all $z \in \mathcal{F}_1$. In either case, the transformation $H$ does not contribute to the fundamental domain $\mathcal{F}$.

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