about a linear fractional transformation of the normal coordinates at every point.

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1 Cf. O. Veblen, these PROCEEDINGS, 8, 1922, p. 192.
2 H. Weyl, Göttinger Nachrichten, 1921, p. 99.
3 A type of normal coordinates invariant under the change (1.1) has been given by O. Veblen and J. M. Thomas in these PROCEEDINGS, 11, 1925, p. 204.
6 G., p. 563.
7 G., p. 562.
8 G., p. 571.
9 G., p. 575, formula (12.1).
10 G., p. 560.
11 G., p. 577.
12 G., p. 579, formula (13.16).
13 G., p. 559, formula (5.10).
14 G., p. 559.
15 In an article which appeared after the present paper had been sent to the printer, J. A. Schouten gives the result contained in the first sentence of this theorem. Cf. Trans. Amer. Math. Soc., 27, 1925, p. 433.

A SIMPLE DERIVATION OF KRONECKER'S RELATION AMONG THE MINORS OF A SYMMETRIC DETERMINANT

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Communicated November 19, 1925

In 1882 Kronecker\(^1\) stated without proof that he had discovered the following theorem:

*Among the minors of order \(m\) of a symmetric system*

\[ a_{ik} = a_{ki} \]

there exists the linear relation

\[ |a_{kh}| = \sum_r |a_{ik}| \]

\((g = 1, 2, \ldots, m; h = m + 1, \ldots, 2m)\)

\((i = 1, 2, \ldots, m - 1, r; k = m + 1, \ldots, r - 1, m, r + 1, \ldots, 2m;\)

\( r = m + 1, m + 2, \ldots, 2m)\).

Numerous proofs and extensions\(^2\) of this important relation have been given since its first announcement. It is the purpose of the present
note to obtain Kronecker’s relation in a very simple manner by the use of certain operators.

Let us first observe that the double operation
\[ \sum_{j=m}^{2m} a_{ij} \frac{\partial}{\partial a_{mj}} \left( \sum_{l=1}^{m} a_{lm} \frac{\partial}{\partial a_{lr}} |a_{eh}| \right) \]
first replaces the elements of column \( r \) of \( |a_{eh}| \) by elements with the same first subscript, but with the second subscript \( m \) and then replaces the elements of the last row of this new determinant by elements with the same second subscript but with the first subscript \( r \). Consequently,
\[ \sum_{r=m+1}^{2m} \sum_{j=m+1}^{2m} a_{ij} \frac{\partial}{\partial a_{mj}} \sum_{l=1}^{m} a_{lm} \frac{\partial}{\partial a_{lr}} |a_{eh}| = \sum_{r} |a_{ik}|, \tag{1} \]
where \( \sum_{r} |a_{ik}| \) is exactly the right member of relation (A).

The left member of (1) is equivalent to
\[ \sum_{r=m+1}^{2m} \sum_{j=m+1}^{2m} a_{ij} \left[ \sum_{l=1}^{m} a_{lm} \frac{\partial}{\partial a_{mj}} \frac{\partial}{\partial a_{lr}} |a_{eh}| \right] + \sum_{r=m+1}^{2m} a_{rm} \frac{\partial}{\partial a_{mr}} |a_{eh}|. \tag{2} \]
Since \( a_{rm} = a_{mr} \), we have
\[ \sum_{r=m+1}^{2m} a_{rm} \frac{\partial}{\partial a_{mr}} |a_{eh}| = |a_{eh}|. \]

The element \( a_{ij} = a_{jr} \ (r > m, j > m) \) occurs linearly and homogeneously in the first part of (2). The coefficient of \( a_{r} = a_{jr} \) vanishes because of the two successive partial derivatives with respect to elements of the same column of \( |a_{eh}| \). The coefficient of \( a_{ij} = a_{jr} \ (r \neq j) \) is
\[ \sum_{l=1}^{m} a_{il} \left( \frac{\partial}{\partial a_{mj}} \frac{\partial}{\partial a_{lr}} |a_{eh}| + \frac{\partial}{\partial a_{mr}} \frac{\partial}{\partial a_{lj}} |a_{eh}| \right). \]
The expression in parenthesis vanishes, a fact which becomes evident if we interchange columns \( r \) and \( j \) in one of the determinants.

The left side of (1) is thus reduced to \( |a_{eh}| \) and we have the relation (A).

Many other relations involving series of determinants may be obtained by the use of operators similar to those employed above. Some of the relations will be given in future papers.

Nanson, E. J., Messenger of Math. (Ser. 2) 31, 1901 (140–143); Amer. J. Math., 27, 1905 (69–76).