the set \( T_1 + T_2 + T_3 + \ldots + T_n \). Let \( K \) denote the set of all points \( X \) of \( M \) such that \( X \) can be joined to \( A \) by a connected subset of \( M \) which contains no point outside of \( C \) and let \( E \) denote the sum of the sets \( H \) and \( K \). Let \( D \) denote the complementary domain of \( E \) which contains \( B \) and let \( L \) denote the boundary of \( D \). Since \( K \) is\(^2\) a continuous curve and each \( T_i \) is\(^3\) a continuous curve, therefore, \( E \) is a continuous curve. Hence \( L \) is\(^3\) a continuous curve.

Now \( A \) does not belong to \( L \). For suppose it does. Then since every point of \( L \) is\(^1\) accessible from \( D \) there exists an arc \( AB \) which, except for \( A \), is a subset of \( D \). Let \( P \) denote the first point that the arc \( AB \) has in common with \( C_1 \). The interval \( AP \) of \( AB \) has only the point \( A \) in common with \( M \) and hence \( AP = A \) lies in one of the complementary domains \( D_1, D_2, D_3, \ldots D_n \). This, however, is impossible since the interval \( BP \) of \( AB \) contains no point of the boundary of any one of these domains. Hence \( A \) lies in one complementary domain of the continuous curve \( L \) and \( B \) lies in another one. Hence by a theorem of R. L. Moore\(^4\) there exists a simple closed curve which separates \( A \) from \( B \) and which is a subset of \( L \) and, therefore, of \( M \).


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RECENT PROGRESS OF INVESTIGATIONS BY SYMBOLICAL METHODS OF THE INVARIANTS OF BI-TERNARY QUANTICS

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The celebrated mathematical discipline, the symbolical theory of invariants of algebraical forms, has its roots in the inspired work of Clebsch. Up to recent times contributions by this method were largely a product of research in Germany.

In the problem determined by forms in two contragredient sets of three variables \((x), (u)\), the object of study is the fundamental system, \( S \), of the connex, \( f = a^m_x a^n_u \). As in the analogous theory of binary forms three objectives may be sought. The first is the method used for the generation of the concomitants, which must be definitive for the formation of all invariants of the existing infinitude. Second, finite expansions analogous to Gordan's series and an appropriate theory of symbolic moduli should be
discovered for use in reducing all invariants in terms of a finite set. Third, the complete system thus found may be investigated from the geometrical point of view, for every such system is the algebraic prototype of a projective geometry of plane loci.

Formation of Invariant Connexes.—If neither exponent \( m, n \) is zero the locus of \( f \) is a connex. In a joint paper\(^1\) published in 1869, in connection with some general symbolical theory of connexes, Clebsch and Gordan determined the complete system of eight concomitants of the bilinear case \( f = a_x \alpha_u \). The subject then remained stationary until 1916 when the present writer published a definitive method\(^2\) of generation for connexes. This is a process of transvection between two forms, analogous to binary transvection as due to Cayley and Gordan. The algorithm is as follows: With two quantics assigned,

\[
\varphi = Aa_1 a_2 x \ldots a_{r_2} a_{r_1} a_{s_2} \ldots a_{s_1},
\]

\[
\psi = Bb_1 b_2 x \ldots b_{r_2} b_{r_1} b_{s_2} \ldots b_{s_1},
\]

(1)

polarize \( \varphi \) by the operator,

\[
\sum (y^{(1)}_1 \frac{\partial}{\partial x})^a (y^{(1)}_2 \frac{\partial}{\partial x})^a \ldots (y^{(1)}_n \frac{\partial}{\partial x})^a (y^{(2)}_1 \frac{\partial}{\partial x})^a (y^{(2)}_2 \frac{\partial}{\partial x})^a \ldots (y^{(2)}_n \frac{\partial}{\partial x})^a,
\]

\[
\times (v^{(1)}_1 \frac{\partial}{\partial u})^a (v^{(1)}_2 \frac{\partial}{\partial u})^a \ldots (v^{(1)}_n \frac{\partial}{\partial u})^a (v^{(2)}_1 \frac{\partial}{\partial u})^a (v^{(2)}_2 \frac{\partial}{\partial u})^a \ldots (v^{(2)}_n \frac{\partial}{\partial u})^a,
\]

where,

\[
\left( y^{(e)}_c \frac{\partial}{\partial x} \right) = y^{(e)}_1 \frac{\partial}{\partial x_1} + y^{(e)}_2 \frac{\partial}{\partial x_2} + y^{(e)}_3 \frac{\partial}{\partial x_3}.
\]

The terms of the summation are uniquely given by the sets of solutions of the restricted system of linear diophantine equations,

\[
\sum_{i=1}^{\sigma} \eta_i = i, \quad \sum_{r=1}^{p} \kappa_r = j, \quad \sum_{r=1}^{p} \lambda_r = k, \quad \sum_{i=1}^{\sigma} \mu_i = l.
\]

These are subject to the conditions,

\( i + j \leq r, \quad k + l \leq s, \quad i + l \leq \sigma, \quad j + k \leq p, \)

and are to be satisfied only by values 0,1 of the variables, \( i, j, k, l \), being fixed chosen integers. With \( t, r \) assigned at least one number in each pair \( (\eta_i, \mu_i), (\kappa_r, \lambda_r), \) must be zero. We substitute in the resulting polar as follows:

\[
\gamma^{(1)}_p = b_p, \quad (p = 1, \ldots, \sigma); \quad \gamma^{(2)}_p = (b_p u), \quad (p = 1, \ldots, \rho),
\]

\[
\nu^{(1)}_p = b_p, \quad (p = 1, \ldots, \rho); \quad \nu^{(2)}_p = (b_p x), \quad (p = 1, \ldots, \sigma).
\]
Finally, multiply each term of the result by the $b, \beta$ factors \((\psi)\), which are not then involved in it. The resulting concomitant is the transvectant,

$$T = (\varphi, \psi)^{\xi}_{\mu}.$$  \((2)\)

If two infinite systems, \([\varphi], [\psi]\), of connexes \((1)\), both have the technical properties of finiteness and completeness, then the system \([C]\) derived by transvection between \([\varphi]\) and \([\psi]\) is finite and complete.

By \([C]\) is meant the system which consists of all terms of all transvectants \(T\), where \(\varphi\) is a form chosen from \([\varphi]\) and \(\psi\) is a form chosen from \([\psi]\). The proof of this theorem gave opportunity for the solution of two problems of enumeration, viz., the simultaneous system of a conic \(F = p^2\) and the connex \(f = a_x\alpha_u\), and the simultaneous system of two connexes \(f = a_x\alpha_u, \varphi = p_x\theta_u\). The system of \(F\) and \(f\) was given in my paper of 1916 and consists of 69 forms. For illustration we give the simplest category of these, that in which the order plus the class of the product of the two forms in a transvectant is 4 \((f_1 = b_x\beta_u a_x, L = (pqu)^3)\).

$$\begin{align*}
(F, f)_{00}^{10} &= p_\alpha p_x a_x, \quad (F, f)_{01}^{01} = (pau)\alpha_u p_x, \quad (F, f)_{10}^{11} \\
(F, f_1)_{00}^{10}, \quad (F, f_1)_{00}^{01}, \quad (F, f_1)_{10}^{11} = (pqa)(pqu)\alpha_u, \\
(L, f)_{00}^{00}, \quad (L, f)_{11}^{01}, \quad (L, f_1)_{00}^{00}, \quad (L, f_1)_{01}^{01}, \quad (L, f_1)_{11}^{00}.
\end{align*}$$

I published the system of the pair \((f, \varphi)\) in 1926. It consists of 193 quantics which were expressed as transvectants.\(^3\)

\textit{Bi-Ternary Reducing Series.—}Several mathematicians have proposed forms of generalization\(^4\) of Gordan's binary series. In a memoir which is soon to be published the present writer has identified a new algorithm in the symbolic theory which gives a reducing series for bi-ternary transvectants, approximating in generality and utility to the binary series\(^5\) of Stroh, for which the claim was made that it would give all syzygetic relations between multiple transvectants on three forms. We shall give an example of this series and the general formula. By a method of contraction which applies to all symbolical monomials we can show that a form,

$$Q = a_c c_p (abc) \gamma_u b_x,$$

is a term of three transvectants, in general, of as many transvectants as there are symbolic pairs \((a, \alpha)\), in which the first form is of degree one. Thus \(Q\) is a term of

$$T = (g, (f, f)_{10}^{01})_{00}^{10}, \quad S = (f, (f, g)_{00}^{01})_{10}^{10},$$

where \(f = a_x^2\alpha_u,\ g = a_x\beta_u\). There is, then, a theorem which was suggested by Gordan\(^6\) but not utilized by him in connection with transvectants, that the difference between a transvectant and one of its terms is a linear combination of transvectants of lower grade number \((\text{cf. } (1)))\), \(N = r +\)
s + ρ + σ, of forms obtained from the original forms (in T) by convolution. Applying this theorem to both T and S we get two values of Q in the form of series, whence,

\[ T - \frac{1}{2} T_1 + \frac{1}{4} T_2u_x + \frac{1}{4} T_3 = S - \frac{1}{2} S_1 + \frac{1}{4} S_2 + \frac{1}{4} S_3, \quad (3) \]

\[ T_1 = (g, c_\beta b(\gamma c x))^{01}_{00}, \quad T_2 = (g, c_\beta (b c \gamma u))^{10}_{00}, \]

\[ T_3 = (g, c_\beta (b c \gamma u))^{00}_{00}, \quad S_1 = (f, a_\alpha (c x))^{01}_{00}, \]

\[ S_2 = (f, a_\alpha (c a \gamma u))^{00}_{00}, \quad S_3 = (g, a_\alpha (c a \gamma u))^{00}_{00}. \]

When these methods are applied in the general case of three arbitrary connexes,

\[ f = a_\alpha x_\delta, \quad g = b_\beta \alpha_\nu, \quad h = c_\xi \gamma_\eta, \]

with

\[ T = (h, f, g)^{ij}_{kl}, \quad \xi = \sigma_1 + s_2 + \tau_2 + t_1, \]

\[ S = (f, (-1)^{\xi} (h, g))^{r^n}_{s^n}, \quad \tau_2 + \tau_3 = s, \quad l_1 + l_2 + l_3 = r, \]

we obtain the series

\[ T + \sum I_1 (h, f, g)^{ij}_{kl} b^{p_1}_{r_1 s_1} u_x^{m_1} \]

\[ = S + \sum I_2 (f, (-1)^{\xi} (h, g))^{r^n}_{s^n} b^{p_1}_{r_1 s_1} d^{q_1}_{r_2 s_2} u_x^{m_2}. \quad (4) \]

The numbers \( I_1, I_2 \) are rational. The form \( P \) indicates application, in a product \( P \), of \( a \) convolutions which replace a product \( p_2 g_\nu \) by \( (p q u) \), \( b \) convolutions which replace \( \alpha_\alpha \beta_\nu \) by \( (a \beta x) \) and \( c \) convolutions which convert \( p_2 \beta_\nu \) into \( p_\eta \). The summations are finite and are arranged according to decreasing grades of the transvectants.

This series adds greatly to our knowledge of the generalities of the subject. Its practical significance is that the series itself and rules of procedure for work of reduction, derived from the principles involved, enable us to find, from the set of fundamental concomitants of degrees \( \leq i-1 \) of \( f = a_\alpha x_\delta \), the complete set of degree \( i \), although the work is complicated in the case of a high degree. We give, in the memoir, a complete system of 32 fundamental concomitants of degrees 1, 2, 3, of \( f = a_\alpha x_\delta \).

Relatively Complete Systems.—The following theorem was proved and applied in obtaining a system of 90 forms for \( f = a_\alpha x_\delta \), relatively finite and complete with respect to a set of symbolical moduli, and also to delimit by theoretical considerations the absolutely finite and complete system. It corresponds to the principal proposition in Gordan’s third proof of Gordan’s theorem.
LEMMA. If a finite system \((A)\) of connexes \(A_1, A_2, \ldots\), all of which are concomitants of \(f = a_x^m a_u^n\), includes \(f\) and is relatively complete for a set \((G)\) of constant symbolic moduli, \(G_1, G_2, \ldots\), and also if a finite system \((B)\) of connexes, \(B_1, B_2, \ldots\), concomitants of \(f\), is relatively complete for a set \((\Gamma)\) of symbolical moduli, \(\Gamma_1, \Gamma_2, \ldots\), and contains the leading form of the form-sequence whose quantics have as their only constant symbolical factor the modulus \(G_i\), for each case \(i = 1, 2, \ldots\), then the system \((C)\) derived by transvection from \((A)\) and \((B)\) is relatively finite and complete modulo \((\Gamma)\).

In reference to geometry a paper\(^7\) by Dr. Sensenig is a branch of this investigation. He used the transvectant \((2)\) in deriving the complete system of the involution of conics \(a_x^2 + kb_x^2\) taken simultaneously with the conic, \((a\beta x)^2\), which is harmonic to \(a_x^2, b_x^2\). The system was reduced in terms of the simultaneous system of \(a_x^2, b_x^2\).

\(^1\) Math. Ann., 1 (1869), 359.
\(^3\) Amer. J. Math., 48 (1926), 45.
\(^6\) Ibid., 1 (1869), 90.
\(^7\) Amer. J. Math., 41 (1919), 111.

A DYNAMICAL THEORY OF ECONOMIC EQUILIBRIUM

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1. Introduction.—In a previous paper I have shown how the dynamic problem of competition is related to a dynamic generalization of the static theory of economic equilibrium of Walras and Pareto.\(^1\) In this paper I intend to simplify the notation of my previous paper and develop the theory of dynamic economic equilibrium for functional equations of demand and supply.\(^2\) I will show that the problem of equilibrium for a cooperative society is a problem for which it is desired to maximize a functional operator, and, also, that the problem for a competitive society is a problem for which it is desired to obtain partial maxima of several functional operators.\(^3\)

Inasmuch as I have already given the economic setting of the problem, I shall not attempt to repeat it in detail.\(^4\) It is sufficient for the purposes of this paper to say that in any economic system certain goods and services are transformed into other goods and services and that we wish to