THE FORM OF THE GROWTH CURVE OF THE CANTELOUP
(CUCUMIS MELO) UNDER FIELD CONDITIONS

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In an earlier paper from this Institute it was shown that the growth of seedlings of the canteloup (Cucumis melo) in the absence of exogenous food and light may be represented by a generalized logistic curve of the form

\[ y = \frac{k}{1 + e^{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}}. \]

The present paper gives the results of observations on the growth of the canteloup plant under normal field conditions. The observations were made on plants grown on our experimental plot during the summer of 1926.

We have used the number of nodes at each observation date as a measure of the growth attained. The relation between number of nodes and length of branch is not strictly linear, but we have found that the departure from linearity does not materially affect the form of the growth curve obtained whichever measure is used. The observations for the largest plant in the group studied are given in table 1. This plant was the one from which were obtained the seeds used in the experiment described in table 2 on page 3 of the paper just referred to. Days are reckoned from July 7, 1926. Some fourteen other plants studied in equal detail give closely similar results to the one regarding which data are presented in this paper. This particular plant (No. 15) is chosen for discussion for the reasons stated above, and because it did not suffer the accidents of the field, such as breaking off of branches in cultivation, injury by insects and rabbits, etc., which to some extent damaged other plants.

In table 1 are also given the number of nodes when classified according to their occurrence on the main stem, primary, secondary, etc., branches. In this subdivision, the main stem and primary branch nodes have been
combined, and also the tertiary and quaternary branch nodes, on account of the small numbers in the main stem and quaternary classes.

### TABLE 1
GROWTH OF CANTHELOUP PLANT, AS MEASURED BY NUMBER OF NODES

<table>
<thead>
<tr>
<th>DAYS FROM JULY 7, 1926</th>
<th>TOTAL NODES OBSERVED</th>
<th>CALCU-LATED</th>
<th>MAIN STEM AND PRIMARY BRANCH NODES OBSERVED</th>
<th>CALCU-LATED</th>
<th>SECONDARY BRANCH NODES OBSERVED</th>
<th>CALCU-LATED</th>
<th>TERTIARY AND QUATERNARY BRANCH NODES OBSERVED</th>
<th>CALCU-LATED</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39</td>
<td>26.8</td>
<td>35</td>
<td>34.1</td>
<td>4</td>
<td>4.0</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>7</td>
<td>69</td>
<td>69.0</td>
<td>54</td>
<td>55.9</td>
<td>15</td>
<td>15.1</td>
<td>0</td>
<td>1.6</td>
</tr>
<tr>
<td>14</td>
<td>151</td>
<td>167.0</td>
<td>92</td>
<td>92.3</td>
<td>55</td>
<td>50.6</td>
<td>4</td>
<td>12.3</td>
</tr>
<tr>
<td>23</td>
<td>370</td>
<td>382.9</td>
<td>152</td>
<td>150.9</td>
<td>139</td>
<td>140.4</td>
<td>79</td>
<td>74.0</td>
</tr>
<tr>
<td>29</td>
<td>572</td>
<td>567.0</td>
<td>192</td>
<td>188.4</td>
<td>224</td>
<td>217.1</td>
<td>156</td>
<td>158.2</td>
</tr>
<tr>
<td>37</td>
<td>842</td>
<td>824.4</td>
<td>223</td>
<td>226.0</td>
<td>321</td>
<td>318.7</td>
<td>298</td>
<td>292.2</td>
</tr>
<tr>
<td>47</td>
<td>1120</td>
<td>1118.2</td>
<td>256</td>
<td>253.7</td>
<td>422</td>
<td>428.4</td>
<td>442</td>
<td>442.0</td>
</tr>
<tr>
<td>51</td>
<td>1211</td>
<td>1220.0</td>
<td>262</td>
<td>259.7</td>
<td>452</td>
<td>464.4</td>
<td>497</td>
<td>496.0</td>
</tr>
<tr>
<td>58</td>
<td>1395</td>
<td>1371.1</td>
<td>265</td>
<td>266.2</td>
<td>520</td>
<td>512.1</td>
<td>610</td>
<td>588.8</td>
</tr>
<tr>
<td>64</td>
<td>1465</td>
<td>1470.1</td>
<td>269</td>
<td>269.3</td>
<td>545</td>
<td>538.1</td>
<td>651</td>
<td>665.0</td>
</tr>
<tr>
<td>71</td>
<td>1549</td>
<td>1549.0</td>
<td>271</td>
<td>271.2</td>
<td>554</td>
<td>551.9</td>
<td>724</td>
<td>736.1</td>
</tr>
<tr>
<td>76</td>
<td>1592</td>
<td>1583.7</td>
<td>271</td>
<td>272.0</td>
<td>554</td>
<td>555.4</td>
<td>767</td>
<td>767.0</td>
</tr>
<tr>
<td>86</td>
<td>1611</td>
<td>1615.6</td>
<td>271</td>
<td>272.8</td>
<td>554</td>
<td>556.9</td>
<td>786</td>
<td>788.1</td>
</tr>
</tbody>
</table>

Equations:

Total nodes

\[
y = \frac{1}{1 + e^{4.264 - 0.180x + 0.00243x^2 - 0.000019x^3}}
\]

Main stem + Primaries

\[
y = \frac{273}{1 + e^{2.045 - 0.098x}}
\]

Secondaries

\[
y = \frac{557}{1 + e^{5.173 - 0.286x + 0.0045x^2 - 0.000059x^3}}
\]

Tertiaries + Quaternaries

\[
y = \frac{790}{1 + e^{8.948 - 0.440x + 0.0071x^2 - 0.000054x^3}}
\]

To the observations as given in the table logistic curves were fitted. The equations of these curves are given above. The diagram shows that the fits are satisfactory.

It is of interest to note that the growth curves become markedly skew as we proceed from the earlier-formed to the later-formed parts of the plant. While the curve for the main stem + primary branch nodes is of the symmetrical type, the secondary branch curve is distinctly skew and the tertiary and quaternary branch curves much more so. In fact, the latter curve is very near the limit of what can be represented without introducing additional points of inflection, if we restrict ourselves to the cubic term in the logistic equation.

If we use the ratio of the ordinate at the point of inflection to the upper
asymptote as a measure of the skewness of the growth curves, we find the following values:

- Main stem + primary: 0.500
- Secondary: 0.463
- Tertiary + quaternary: 0.285

In the present case the curve representing the total nodes is a skew logistic. This is not always the case; in some of our plants the symmetrical gives a satisfactory fit to the observations of total nodes. In these cases, however, it remains true that the component curves show progressively greater skewness in later-formed parts of the plant.

The fact that the sum of a number of generalized logistics can be adequately represented by a logistic is of some interest. Reed and Pearl have recently studied the summation of symmetrical logistics and their representation by a single logistic, of either the symmetrical or skew type, and have developed a mathematical theory for this case. The data presented in this paper indicate that in the more general case of skew logistics, a summation into a single logistic is also possible.
The theoretical significance of the investigation reported in this paper lies in two directions. In the first place the observations complete the demonstration of the conclusion reached in our former paper, that the form of the growth curve is logistic when growth is the expression either of what we have called "inherent vitality" or of "total vitality." In the earlier paper it was shown that seedlings of the canteloup grown under conditions such that all vital activity was necessarily an expression solely of the inherent organization, constitution, or organic pattern of the plant (Situation B) grew according to a logistic curve. In the present paper it is shown that the identical plant from which came the seeds used in the seedling experiment itself grew according to a logistic curve under normal field conditions (Situation A), in which conditions growth was an expression of total vitality, as defined in earlier contributions from this Institute.

The two logistic curves concerned in this case may be put on a directly
comparable basis quite simply by expressing growth attained at any
given time as percentage of the asymptotes in the two cases, and by
expressing time as percentage of the total active growing period. The
relative values so obtained are shown in table 2. In this table the part
of Plant No. 15 chosen for comparison with its own seedlings is the main
stem + primary branches. The seedling grown under Situation B condi-
tions is morphologically an unbranched stem with roots at the bottom
and cotyledons at the top. Of the morphological categories set forth
in table 1 above, the second, main stem + primary branches, comes
closest to the seedling.

| TABLE 2 | RELATIVE GROWTH OF SEEDLINGS WITHOUT EXOGENOUS FOOD OR LIGHT (SITUATION B) AND OF THE MAIN STEM + PRIMARY BRANCHES OF THE PLANT FROM WHICH THESE SEEDS CAME, GROWING UNDER NORMAL FIELD CONDITIONS (SITUATION A) |
|---------------------------------|---------------------------------|---------------------------------|
| SEEDLINGS FROM PLANT 15         | MAIN STEM + PRIMARY BRANCHES    | PLANT 15                        |
| TIME AS PER CENT OF ACTIVE GROWING PERIOD | OBSERVED HEIGHT AS PER CENT OF ASYMPTOTE | TIME AS PER CENT OF ACTIVE GROWING PERIOD | OBSERVED HEIGHT AS PER CENT OF ASYMPTOTE |
| 0                               | 1.9                             | 0                               | 12.8                             |
| 7.3                             | 3.1                             | 8.6                             | 19.8                             |
| 13.5                            | 13.7                            | 18.6                            | 33.7                             |
| 21.4                            | 34.5                            | 31.4                            | 55.7                             |
| 37.7                            | 69.6                            | 40.0                            | 70.3                             |
| 45.3                            | 79.2                            | 51.4                            | 81.7                             |
| 53.1                            | 87.6                            | 65.7                            | 93.8                             |
| 61.0                            | 93.3                            | 71.4                            | 96.0                             |
| 70.0                            | 97.3                            | 81.4                            | 97.1                             |
| 76.3                            | 98.5                            | 90.0                            | 98.5                             |
| 91.9                            | 99.6                            | 100.0                           | 99.3                             |

If the data of table 2 are plotted graphically the result shown in figure 2
is obtained.

It is at once apparent that when the two growth curves are reduced to
relative terms, in respect of both ordinate and abscissa, there is excellent
agreement between them. The only noteworthy discrepancy is at the
beginnings of the curves. This is mainly, if not entirely, due to a technical
observational difficulty, which we have not been able to overcome. In
the case of both the whole plant in the field, and the seedling in the sterile
tube a good deal of growth takes place before it is possible to measure it.
The field plant must get above ground and grow enough thereafter to
separate a few nodes, at least, before they can be observed and counted.
Similarly the seed in the tube must germinate, and get oriented relative
to gravity and start a stem upward before it can be measured. This
delay in observations is greater for the field plant than for the seedlings.

The second theoretical consideration derivable from the experiments
to which we wish to call attention may be developed as follows. If we
analyze, in terms of the most complete abstractions permissible in the premises, what we have called above Situation A and Situation B the case comes to this.

Situation A involves three distinguishable elements, viz.:

(a) *The organism*, which is an integrated pattern, or in Whitehead's words "a general organic plan," through which constantly flows a stream of

(b) *Extraneous matter and energy*, falling into three categories; (1) (H₂O and O₂), (2) (various compounds of Na, K, Ca, Mg, P, S, Fe, Cl, Si, C), and (3) (radiant energy from sunlight), all of which come from the

(c) *Environment*, which varies greatly with time throughout the life history of the plant in all physical and chemical characteristics, and in which the organism (a) also has its being.

Situation B, on the other hand, involves also three distinguishable elements, viz.:

(a) *The organism*, through which passes

(b) *Extraneous matter*, falling, however, only in one category (1) (H₂O and O₂), derived from an

(c) *Environment* which is constant throughout the life history of the plant in all physical and chemical particulars except those relating to (H₂O and O₂), using the word constant to mean, precisely, less variable to an indefinite degree than the environment in Situation A.

Situation B therefore differs from Situation A in the following respects:

1. The absence of elements (2) and (3) in item (b), which are present in Situation A.

2. The constancy of (c), as contrasted with its variance in Situation A.

The organism in Situation B exhibits and goes through all the phases of the life cycle (growth, stable continuance of life after growth, senescence, death) which are exhibited by the organism in Situation A. The duration of each of these phases in Situation B is curtailed in respect of absolute time, but agrees, to a first approximation, in respect of relative time, with the duration of the corresponding phases in Situation A. Whence it is obvious that those elements of Situation A which are absent in Situation B are to be regarded as essential to life only in respect of its duration in absolute time. They are simply means of prolonging in time the manifestations of the vital potential of the organism, which is inherent in its pattern or organization. They are fuel which keeps the machine running.

From the point of view of theoretical genetics, it is of considerable importance to note that the great variance in the environment (c) in Situation A as contrasted with its constancy in Situation B is without
effect on the form, quantitatively considered, of one highly important phase of the life cycle, namely, growth. More precisely stated, the relation between attained size and relative time in the development of the canteloup plant, is, to a first approximation, identical whether the environment in which it has its being is highly variable in respect of temperature and all other physical and chemical particulars, or is, to a high degree, constant. The pattern of the events which constitute the life cycle of the organism is primarily and basically determined by the physico-chemical organization or pattern of the organism itself.

1 From the Institute for Biological Research of the Johns Hopkins University.

INVARIANT SEQUENCES

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1. Introduction.—Let \( f_n(x) (n = 0, 1, \ldots) \) be uniform functions of the real or complex variable \( x \); let \( a, b, \tau (n) \) be independent of \( x \). Then if \( a, b, \tau (n) \) exist such that

\[
    f'_n(x) = f_{n-1}(x), \quad f_n(ax + b) = \tau (n) f_n(x), \quad \left[ f'_n(x) = \frac{d}{dx}f_n(x) \right],
\]

for \( n = 0, 1, 2, \ldots \), we shall say that the invariant sequence \( f_n(x) (n = 0, 1, \ldots) \) has the characteristic \( (a, b, \tau(n)) \). It is known that the general solution of the first of the above functional equations is \( f_n(x) = k(x + \varphi)^n/n! \), where \( \varphi \) (including \( n = 0 \)) is to be replaced by \( \varphi^n \) after expansion by the binomial theorem, and \( k \) is independent of both \( n \) and \( x \). Hence the \( n \)th element of an invariant sequence is a polynomial of degree \( n \). We call \( \varphi \) as above defined the base of \( f_n(x) (n = 0, 1, \ldots) \), and, if \( s \geq 0 \) is the least integer such that \( \varphi_s \neq 0 \), we say that \( s \) is the index of \( \varphi \). We regard \( \varphi \) as the quaesitum. In terms of \( \varphi, s, n \) we shall find \( a, b \) and \( \tau(n) \).