On Einstein's Unified Field Equations and the Schwarzschild Solution

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In recent papers Wiener and the author have determined the tensors \( h_x \) of Einstein's unified theory of electricity and gravitation under the assumptions of static spherical symmetry and of symmetry of past and future. It was there shown that the field equations suggested in Einstein's second 1928 paper lead in this case to a vanishing gravitational field. The purpose of this paper is to investigate, for the same case, the nature of the gravitational field obtained from the field equations suggested by Einstein in his first 1929 paper. These field equations are
\[ h(\varphi_{\mu}; \delta - \varphi_{\mu}; \alpha) /\alpha = 0 \] \hspace{1cm} (1)
\[ \bar{V}^{\alpha}_{\beta\gamma} - \bar{V}^{\alpha}_{\beta\gamma} \Lambda^\alpha_{\gamma\nu} = 0. \] \hspace{1cm} (2)

A word of explanation of the symbolism found here is perhaps necessary. The tensors appearing in the formulas are defined as follows:

\[ \Lambda^\alpha_{\mu} = \Gamma^\alpha_{\mu
u} - \Gamma^\mu_{\nu\alpha}, \quad \Gamma^\mu_{\nu\alpha} = s h^\alpha_{\nu} \frac{\partial h^\mu}{\partial x^\alpha}, \]
\[ \varphi_a = \Lambda^a_{\alpha\mu}, \quad h = |s h|, \]
\[ V^\alpha_{\nu\mu} = \Lambda^a_{\nu\mu} + \varphi_a \delta^a_{\mu} - \varphi_b \delta^a_{\nu}. \]

\& is Kronecker's symbol; a semicolon indicates affine covariant differentiation corresponding to \( \Gamma^\mu_{\nu\alpha} \), and a bar over a tensor indicates the corresponding tensor density (e.g., \( \bar{T} :: = h T :: \)). An underlined index indicates that the index in question is to be raised (or lowered), and the symbol /\( \alpha \) denotes the operation closely akin to divergence

\[ \bar{T} :: /\alpha = h T ::/\alpha - \bar{T} ::/\alpha \Lambda^\alpha_{\nu\mu}. \] \hspace{1cm} (3)

The expression \( \varphi_{\mu}; \alpha - \varphi_{\alpha}; \mu \) which according to Einstein defines the electromagnetic field tensor, has the value

\[ E_{\mu\alpha} = \varphi_{\mu}; \alpha - \varphi_{\alpha}; \mu = \frac{\partial \varphi_{\mu}}{\partial x^\alpha} - \frac{\partial \varphi_{\alpha}}{\partial x^\mu} + \Lambda^\mu_{\alpha\nu} \varphi_{\nu} = F_{\mu\alpha} + \Lambda^\mu_{\alpha\nu} \varphi_{\nu}, \]

\( F_{\mu\alpha} \) is the ordinary electromagnetic tensor. Equation (1) may therefore be written

\[ \bar{E}_{\mu\alpha}/\alpha = 0. \] \hspace{1cm} (1a)

We also have

\[ h V^\alpha_{\beta\gamma}; \gamma = \bar{V}^\alpha_{\beta\gamma}; \gamma = h \frac{\partial V_{\alpha\beta\gamma}}{\partial x^\gamma} + \Gamma^\alpha_{\sigma\gamma} V^\sigma_{\nu\beta\gamma} + \Gamma^\mu_{\gamma\beta} V^\nu_{\sigma\tau\gamma} + \Gamma^\gamma_{\sigma\tau} V^\tau_{\nu\beta\gamma}, \]

therefore, from (3), Equation (2) may be written

\[ h \frac{\partial V_{\alpha\beta\gamma}}{\partial x^\gamma} + V_{\alpha\beta\tau} \Gamma^\tau_{\nu\sigma\gamma} + V_{\alpha\sigma\tau} \Gamma^\tau_{\nu\beta\gamma} - V_{\alpha\beta\gamma} \Lambda^\gamma_{\nu\sigma} - V_{\nu\beta\gamma} \Lambda^\gamma_{\sigma\tau} = 0; \]

or simplifying and dividing through by \( h \)

\[ \frac{\partial V_{\alpha\beta\gamma}}{\partial x^\gamma} + V_{\nu\beta\gamma} \Gamma^\nu_{\sigma\tau\gamma} + V_{\alpha\sigma\tau} \Gamma^\tau_{\nu\beta\gamma} + V_{\alpha\beta\gamma} \Gamma^\tau_{\nu\sigma\gamma} = 0. \] \hspace{1cm} (2a)

In accordance with the assumptions of static spherical symmetry and symmetry of past and future the line element is taken to be

\[ ds^2 = U^2(r) dr^2 + r^2 \sin^2 \theta d\phi^2 + r^2 d\theta^2 + W^2(r) dt^2. \]

Using Cartesian coordinates in the local quadruples and Cartesian (Gaussian) coordinates.
in the space-time continuum, it is readily found by availing ourselves of the results of our second paper (loc. cit.) that there are 18 non-vanishing components of the tensor $\Lambda_{\mu\nu}$ as follows:

\[
\begin{align*}
\Lambda_{12} & = -\Lambda_{21} = -\Lambda_{33} = \Lambda_{32} = -Ay \\
\Lambda_{13} & = -\Lambda_{31} = -\Lambda_{22} = \Lambda_{23} = -Az \\
\Lambda_{14} & = -\Lambda_{41} = -xW'/W r \\
\Lambda_{24} & = -\Lambda_{42} = -yW'/W r \\
\Lambda_{34} & = -\Lambda_{43} = -zW'/W r \\
\end{align*}
\]

with the abbreviations $A = (u - 1)/r^2$ and $W' = \partial W/\partial r$. The components of the electromagnetic potential $\phi_\mu$ are, therefore, as already noted before [loc. cit. (1), second paper]

\[
\phi_1 = 2Ax, \quad \phi_2 = 2Ay, \quad \phi_3 = 2Az, \quad \phi_4 = 0.
\]

The non-zero components of the tensor $V^\mu_\nu$, are

\[
\begin{align*}
V^1_{12} & = -V^1_{21} = -V^2_{33} = V^3_{23} = Ay \\
V^2_{12} & = V^3_{13} = -V^2_{31} = -V^3_{21} = -Ax \\
V^1_{13} & = V^2_{23} = -V^2_{32} = -V^1_{31} = Az \\
V^4_{14} & = -V^4_{41} = -x \left( \frac{W'}{W r} + 2A \right) \\
V^4_{24} & = -V^4_{42} = -y \left( \frac{W'}{W r} + 2A \right) \\
V^4_{34} & = -V^4_{43} = -z \left( \frac{W'}{W r} + 2A \right),
\end{align*}
\]

and of the contravariant tensor $V^{\alpha\beta\gamma}$

\[
\begin{align*}
V^{112} & = -V^{121} = -V^{333} = V^{332} = Ay/U^2 \\
V^{113} & = -V^{131} = V^{233} = -V^{332} = Az/U^2 \\
V^{114} & = -V^{141} = -x \frac{W'}{U^3 W^2} \left( \frac{W'}{W r} + 2A \right) \\
V^{424} & = -V^{442} = -y \frac{W'}{U^3 W^2} \left( \frac{W'}{W r} + 2A \right) \\
V^{434} & = -V^{443} = -z \frac{W'}{U^3 W^2} \left( \frac{W'}{W r} + 2A \right).
\end{align*}
\]

We now consider the "electromagnetic" equation (1), or (1a). We obtain in this case for all possible combinations of $\mu$ and $\alpha$

\[ E_{\mu\alpha} = 0. \]

Equation (1) reduces, therefore, to a trivial identity. The vanishing of the electrostatic field, already emphasized in our previous papers, is of
course a consequence of the definition of the electromagnetic potential and is independent of the particular choice of the field equations.

We now turn our attention to the "gravitational" equations (2a). The free indices are here $\alpha$ and $\beta$, and we have the following five possible types of equations: $\alpha = 1, \beta = 1$; $\alpha = 1, \beta = 2$; $\alpha = 1, \beta = 4$; $\alpha = 4, \beta = 1$; $\alpha = 4, \beta = 4$. Of these, two reduce to trivial identities, i.e., for $\alpha = 1, \beta = 4$ and for $\alpha = 4, \beta = 1$. The equation corresponding to $\alpha = 1, \beta = 1$ is

$$\frac{\partial V_{11}}{\partial y} + \frac{\partial V_{11}}{\partial z} + (2\Gamma_{12}^{1} + \Gamma_{22}^{2} + \Gamma_{22}^{4} + \Gamma_{42}^{4}) V_{11} + (2\Gamma_{13}^{1} + \Gamma_{23}^{2} + \Gamma_{33}^{3} + \Gamma_{43}^{4}) V_{11} + \Gamma_{13}^{1} V_{21}^{2} + \Gamma_{13}^{1} V_{31}^{1} = 0,$$

which on substituting the values of the different quantities and performing the indicated operations yields

$$\frac{2}{U^2} - \frac{2}{U^3} + \frac{2}{U^2} \frac{U'}{U^3} - \frac{2}{U^2} + \frac{2}{U^4} - \frac{2}{U^2} U'(U - 1) + \frac{U - 1}{r^2 U} \left( \frac{U'}{U^3} + \frac{W'}{W^3} \right) - \frac{2}{U^2} \frac{U'}{U^3} \left( \frac{1}{U^2} - \frac{1}{U^3} \right) (r^2 - x^2 - z^2) = 0.$$

Likewise for $\alpha = 2, \beta = 2$ and for $\alpha = 3, \beta = 3$ we obtain the analogous equations

$$\frac{2}{U^2} - \frac{2}{U^3} + \frac{x^2 + z^2}{U^2} \left( \frac{U'}{U^2} - \frac{2}{U^4} + \frac{2}{U^4} \right) - \frac{2}{U^2} \frac{x^2 + z^2}{U^2} U'(U - 1) + \frac{U - 1}{r^2 U} \left( \frac{U'}{U^2} + \frac{W'}{W^2} \right) - \frac{2}{U^2} \frac{U'}{U^2} \left( \frac{1}{U^2} - \frac{1}{U^3} \right) (r^2 - x^2 - z^2) = 0.$$

These three equations reduce to the following pair

$$\frac{1}{U} = 1$$

$$\frac{1}{U} \left( \frac{U'}{r} - \frac{2}{r^2} \right) = \frac{2}{U^2} \left( \frac{1}{r^2} - \frac{1}{U^2} \right) = 0.$$

the only solution of which is $U = 1$. 

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The equation belonging to the type corresponding to $\alpha = 1$, $\beta = 2$ is
\[
\frac{\partial V^{121}}{\partial x} + V^{112}A^5_{12} + (2\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31} + \Gamma^4_{41})V^{121} + \Gamma^1_{12}V^{221} + \Gamma^1_{22}V^{223} + \Gamma^1_{33}V^{323} = 0,
\]
which gives
\[
- \frac{U'}{U} - \frac{UW'}{W} + \frac{W'}{W} - \frac{U}{r} - \frac{2}{Ur} + \frac{3}{r} = 0.
\]
The solution obtained above is immediately seen to be consistent with this equation.

The equation belonging to the type $\alpha = 4$, $\beta = 4$ is
\[
\frac{\partial V^{441}}{\partial x} + \frac{\partial V^{442}}{\partial y} + \frac{\partial V^{443}}{\partial z} + \Gamma^4_{41}V^{441} + \Gamma^4_{42}V^{442} + \Gamma^4_{43}V^{443} + (\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31} + \Gamma^4_{41})V^{441} + (\Gamma^1_{12} + \Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42})V^{442} + (\Gamma^1_{13} + \Gamma^2_{23} + \Gamma^3_{33} + \Gamma^4_{43})V^{443} = 0,
\]
which reduces to
\[
W'' + \frac{2W'}{r} - \frac{W''}{W} = 0,
\]
which is not the Schwarzschild equation for $W$. It is therefore seen that the field equations suggested by Einstein in his first paper of 1929 do not yield the Schwarzschild solution. While space-time is not flat, space alone is Euclidean.

It may be mentioned in closing that Wiener has shown in a paper to be published elsewhere soon that the Schwarzschild solution satisfies exactly the field equations suggested by Einstein in his second 1929 paper. This Wiener proves using precisely the same tensors of this paper. Indeed, as he has shown, these latest field equations reduce identically to the equations of the gravitational theory of 1916.


