THE CRITICAL POINTS OF A FUNCTION OF n VARIABLES

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This paper contains among other results a treatment of the critical points of a real analytic function without restriction as to the nature of the critical points. Together with the author's unpublished results on the removal of the boundary conditions it constitutes a complete treatment of the problem, the first of its kind.

In the most of the paper the function considered is of class $C''$, and may have critical loci not even complexes, in fact, an infinite set of such loci. Moreover, even in the analytic case it is not assumed that the critical loci are complexes, a considerable advantage.

Let $(x)$ be a point in Euclidian $n$-space in a finite region $\Sigma$, bounded by a closed set $M$, consisting of a finite number of connected, regular, non-intersecting $(n-1)$ spreads of class $C''$. Let $f(x)$ be a real function of $(x)$ of class $C''$ on $\Sigma$, with a positive, exterior normal directional derivative on $M$. We assume the critical values of $f$ are finite in number.

By a critical set $g$ will be understood any closed set of critical points of $f$ which yields one critical value, which includes all critical points to which it may be connected among critical points, and which is at a positive distance from other critical points. It may or may not be connected. In general, it will not be a complex. Suppose we have a critical set at which $f = 0$.

Let $N$ and $N'$ be any two open neighborhoods of $g$ of which $N^1$ is interior to $N$. By $\overline{N}$ and $\overline{N'}$ we shall mean those points of $N$ and $N'$ at which $f < 0$.

In the analytic case, and much more generally the following sets of cycles exist for an $N^1$ sufficiently near $g$.

$(a)$. A complete set $(a)_k$ of $k$-cycles on $\overline{N}$, independent on $\overline{N}$, dependent on $N$.

$(b)$. A complete set $(c)_k$ of $k$-cycles on $N^1$ independent on $N$ of the $k$-cycles on $\overline{N}$. (Called critical cycles.)

Moreover, for neighborhoods $N$ sufficiently near $g$ and correspondingly small $N^1$, the number of cycles in these sets is independent of $N$ and $N^1$. This we prove for cases much more general than the analytic.

We define the $k$th type number $M_k$ of a critical set $g$ to be the number of cycles in sets $(a)_{k-1}$ and $(c)_1$.

From its definition it appears that the type numbers are invariant under homeomorphisms that carry critical sets into critical sets. The type number of a finite ensemble of critical sets is the sum of the type numbers of the component sets.

We replace $(a)_{k-1}$ by an algebraically equivalent set composed of two
sets, a set $(l)_{k-1}$ of $(k-1)$-cycles that bound on $f < 0$, called linkable, and a set $(b)_{k-1}$ called newly bounding, no linear combination of whose members, not null, so bounds. Each member of $(l)_{k-1}$ bounds a $k$-chain on $f < 0$, and one on $N$. The difference of these two $k$-chains gives a $k$-cycle linking the member of $(l)_{k-1}$.

Let $a$ and $b$ be two ordinary values of $f$ between which lies just one critical value zero, with a critical set $g$, not necessarily finite, or connected, or a complex. Let the domains $f \leq a$ and $f \leq b$ be denoted, respectively, by $A$ and $B$. Let us call a $k$-cycle, on $A$, which is non-bounding on $B$, invariant.

We have the following theorem whose proof, largely by means of deformations, takes less than two pages.

**Theorem.** A complete set of $k$-cycles for $B$ is composed of complete sets of invariant, linking, and critical $k$-cycles, respectively (Mod. $m$ Prime).

We associate with $g$, $m^*_k$ and $m_k$ ideal critical points of increasing and decreasing types, respectively, of which $m^*_k$ shall equal the number of linking and critical $k$-cycles in complete sets, and of which $m_k$ shall equal the number of newly bounding $(k-1)$-cycles in a complete set.

Let $a$ and $b$ be any two non-critical constants. Between the differences $\Delta R_k$, namely, the $k$th Betti number of $B$ minus that of $A$, and the numbers $M_k, m^*_k, m_k$, summed over all critical sets on $B - A$ the earlier relations now hold, namely,

$$\Delta R_k = m^*_k - m^*_{k+1} \quad (k = 0, 1, \ldots, n)$$

$$M_k = m^*_k + m_k$$

with $m^*_0 = m^*_{n+1} = 0$. Important equalities are obtained by eliminating the numbers $m_k$. Similarly for $m^*_k$. A host of equalities and inequalities results.

We also prove that $M_k$ will exceed or equal the number of $k$-cycles on $B$, independent of $k$-cycles on $A$, in a complete set, plus the number of $(k-1)$-cycles on $A$ independent on $A$, but bounding on $B$, in a complete set.

We determine the type numbers more simply in terms of neighborhood functions $\varphi(x)$ related to each critical set $g$.

Such a function $\varphi$ shall be of class $C^\infty$ neighboring $g$, take on a proper relative minimum on $g$, and possess a gradient never parallel to that of $f$ at points not on $g$ at which $f = 0$.

We show that $\Sigma f_x f_x$ is always a neighborhood function in the analytic case. For maximizing or minimizing critical sets, $-f$ and $f$ serve respectively as neighborhood functions in any case whatsoever. We prove the existence of functions $\varphi$ in other cases.

Let $\varphi_\varepsilon$ and $\varphi^*_\varepsilon$, respectively, denote the domains $\varphi = \varepsilon$ and $0 < \varphi \leq \varepsilon$, where $\varepsilon$ is a positive constant less than a certain positive constant $r$. The points of these domains at which $f < 0$ will be respectively denoted by $\varphi_\varepsilon$ and $\varphi^*_\varepsilon$. The domain $\varphi \leq \varepsilon$ will be denoted by $\varphi^*$. 
We show that the domains \( \varphi_e \) are bounded by non-singular, closed, regular \((n-2)\)-spreads, and are homeomorphic for each \( e \). Each domain \( \varphi_e \) is deformable on itself without increasing \( \varphi \) onto any domain \( \varphi_\eta \) with \( \eta < e \), keeping \( \varphi_e \) fixed. The domains \( \varphi_e \) are homeomorphic for each \( e \).

The type number \( M_k \) is shown to be the number of cycles in the following complete sets.

(a) A complete set of \((k-1)\)-cycles on \( \varphi_e \) independent on \( \varphi_e \), bounding on \( \varphi^e \).

(b) A complete set of \( k \)-cycles on \( \varphi_e \) independent on \( \varphi_e \) of the cycles on \( \varphi_e \) (critical \( k \)-cycles).

We see that \( M_0 \) and \( M_n \) are null except when the critical set is respectively minimizing or maximizing, and in the later cases one for connected sets.

For a minimizing set on which \( f = 0 \) we can take \( \phi = f \). We see that (a) is empty, and \( M_k \) is the \( k \)th Betti number of \( f \leq e \) neighboring \( g \). For a maximizing set we take \( \varphi = -f \). We see that (b) is empty in Euclidean \( n \)-space and \( M_k = R_{n-k} \), where \( R_{n-k} \) is a Betti number of \( f \geq -e \).

In the case of an isolated critical point, \( f \) analytic, we can take \( \varphi = x_i x_i \) and Brown's definition is given by (a) and (b), so that his results on critical points all follow. The author's earlier results are, with more difficulty, included in this paper. Brown's results* in the plane also follow.

We justify our results further by showing that we can deform the function slightly, preserving boundary conditions and class, obtaining a function with non-degenerate critical points which are grouped in the respective neighborhoods of the critical sets of \( f \). Moreover, in the neighborhood of each connected critical set of \( f \) the new critical points there found have type number sums at least as great as the corresponding type number of \( g \).

In the analytic case the deformation is made analytically.

The author has just finished a proof that the type numbers of a general critical set always exist, and are finite. The critical values are assumed isolated only for points neighboring the given critical set.

* Brown, Ann. Math., 31 (1930), p. 449. Here also are references to Birkhoff, Whyburn, Poincaré, Brown, Morse, and for analysis situs Alexander. See also


Morse, Trans. Amer. Math. Soc., 36 (1930). In this paper is found the origin of the methods of the present paper.

The Colloquium Lectures by Lefchetz, not yet out at the time of writing, will undoubtedly be of value in this connection.