
Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914, p. 213.

If \( K \) is a set of points, by \( K' \) we denote the set consisting of \( K \) together with its limit points.

These Proceedings, 12, 1926 (761–767).

Compare the first part of this proof with that given by Whyburn for the case cited above.

Wilder, R. L., Bull. Amer. Math. Soc., 32, 1926 (338–340). It is clear from its proof that this theorem is true in any topological space.


Wilder, R. L., Bull. Amer. Math. Soc., 34, 1928 (649–655). The theorem quoted here was proved for euclidean spaces, but it is clear that it holds in any locally compact metric space, and, indeed, may be proved directly by use of the imbedding theorem employed in proving Theorem 4 below, and hence so that \( N \) is of the same dimension (Menger-Urysohn) as \( H \), etc.


Wilder, R. L., these Proceedings, 11, 1925 (725–728). Although the proof is given in the terminology of the plane, it is clear that the theorem referred to is true in any locally compact metric space; in fact, the necessity proof holds in any topological space.

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COMBINATORY TOPOLOGY OF CONVEX REGIONS

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In a recent paper\(^1\) I have shown that a set of rectilinear simplexes filling a convex region in \( R_n \) (Euclidean \( n \)-space) form an \( E_n \), i.e., an \( n \)-element in the combinatorial sense. The first part of the proof there given may be replaced by the considerably shorter argument which follows. For the idea of this simplification I am indebted to Professor J. W. Alexander.

The theorem to be proved is as follows:

If \( A_n \) is a finite collection of (closed) rectilinear \( n \)-simplexes in \( R_n \) satisfying the conditions: (1) any two simplexes have either nothing or a \( k \)-component in common \((0 \leq k \leq n - 1)\); (2) the points of \( A_n \) form a convex region in \( R_n \); then the set of simplexes form an \( E_n \).

Consider a convex region divided into convex rectilinear \( n \)-cells of any type, where by a convex rectilinear \( n \)-cell is meant a closed bounded region cut out of \( R_{n + 1} \) by a finite number of flat \((n - 1)\)-folds. A set of simplexes satisfying \( R_1 \) and \( R_2 \) can be obtained from such a collection of
cells by regular subdivision, i.e., by replacing the $1$-, $2$-, \ldots, \,$n$-cells successively by $1$-, $2$-, \ldots, \,$n$-starchs of simplex. An argument used in the paper referred to\textsuperscript{2} shows that if it could be proved that regular subdivision of any such convex collection of convex cells (in particular of $A_n$), leads to an $E_n$, it would follow that $A_n$ itself is an $E_n$. We wish then to prove

**Theorem A.** If $K_n$ is a set of convex rectilinear $n$-cells filling a convex region in $R_n$ in such a way that the common part of two cells is either nothing or a $k$-cell of both $(0 \leq k \leq n-1)$, then the set of simplices obtained by regular subdivision of $K_n$ is an $E_n$.

The theorem being trivial if $n=1$, suppose it true of sets of cells in $R_{n-1}$.

We can derive from $K_n$ another collection of cells $K_n^*$, of a more special type, by "producing" all the $(n - 1)$-dimensional faces, i.e., introducing as barriers the complete intersection of $K_n$ with all flat $(n - 1)$-folds containing an $(n - 1)$-face of an $n$-cell. This has the effect of replacing each $n$-cell of $K_n$ by a collection of $n$-cells of the type of $K_n$ itself. By the argument already referred to\textsuperscript{2} if Theorem A can be proved for $K_n^*$ its truth follows for $K_n$.

$K_n^*$ has these properties: ($P_1$) no two boundary $(n - 1)$-cells of the same $n$-cell lie in the same flat $(n - 1)$-fold, ($P_2$) if a flat $(n - 1)$-fold contains an $(n - 1)$-cell all its points in $K_n^*$ belong to $(n - 1)$-cells.

(a) Suppose $K_n^*$ consists of a single cell. Let $C_{n-1}$ be any boundary $(n - 1)$-cell, $\Pi_{n-1}$ the sum of the remaining $(n - 1)$-cells,\textsuperscript{3} $\pi_{n-1}$ the $(n - 1)$-fold containing $C_{n-1}$. Let $V$ be a point separated from the interior of $K_n^*$ by $\pi_{n-1}$, but so near the center of gravity of $C_{n-1}$ that if a straight line through $V$ meets $K_n^*$ it meets $C_{n-1}$. Since $K_n^*$ is convex it follows that every line through $V$ which contains a point of $K_n^*$ contains one point of $C_{n-1}$ and one point of $\Pi_{n-1}$. Thus projection of $\Pi_{n-1}$ on to $\pi_{n-1}$ from $V$ gives a $(1,1)$-continuous representation of $\Pi_{n-1}$ as a set of convex rectilinear $(n - 1)$-cells filling the convex region $\tilde{C}_{n-1}$. The conditions of Theorem A are satisfied (with $n - 1$ for $n$) and $\Pi_{n-1}$, on regular subdivision, becomes an $E_{n-1}$. Hence, by a known theorem\textsuperscript{4} $C_{n-1} + \Pi_{n-1}$, the complete boundary of $K_n^*$, becomes on regular subdivision a $\Sigma_{n-1}$—a combinatorial $(n - 1)$-sphere. It now follows\textsuperscript{4} that regular subdivision of $K_n^*$—the join of a central vertex to a $\Sigma_{n-1}$—gives an $E_n$.

(b) The proof for any set of cells $K_n^*$ satisfying ($P_1$) and ($P_2$) now proceeds by induction through the number of cells. Let $\rho_{n-1}$ be any $(n - 1)$-fold containing an $(n - 1)$-cell interior to $K_n^*$. No cell of $K_n^*$ is divided in two by $\rho_{n-1}$: two cells lying on its two sides form two collections of $n$-cells satisfying ($P_1$) and ($P_2$), which by an inductive hypothesis, may be assumed to become two $E_n$'s on regular subdivision. Their common points form a set of $(n - 1)$-cells, satisfying ($P_1$) and ($P_2$), i.e., on regular subdivision they form an $E_{n-1}$. Hence\textsuperscript{5} the sum of the two $E_n$'s is itself an $E_n$. 


\textsuperscript{3} See footnote \textsuperscript{2}.


This completes the proof of Theorem A.


2 Loc. cit., p. 343, par. (d).

3 All simplexes and cells in this paper are closed sets.

4 FII, Theorem 6, Coroll. 1.

5 FII, 12.

6 FII, Theorem 8a, Coroll.

SOME REMARKS ON THE PROBLEM OF PLATEAU

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1. The following remarks are concerned with a recent paper of R. Garnier on the problem of Plateau.¹ This problem, as investigated by Garnier according to the ideas of Riemann, Weierstrass and Schwartz, may be stated in the following analytic form:

Given, in the xyz-space, a simple closed curve \( \Gamma^* \). Determine three functions \( x(u, v), y(u, v), z(u, v) \) with the following properties.

I. \( x(u, v), y(u, v), z(u, v) \) are harmonic for \( u^2 + v^2 < 1 \)

II. Satisfy for \( u^2 + v^2 < 1 \) the relations

\[
\begin{align*}
  x_u^2 + y_u^2 + z_u^2 &= x_v^2 + y_v^2 + z_v^2, \\
  x_u x_v + y_u y_v + z_u z_v &= 0.
\end{align*}
\]

III. \( x(u, v), y(u, v), z(u, v) \) remain continuous on the unit circle \( u^2 + v^2 = 1 \) and carry the unit circle in a one-to-one and continuous way into the given curve \( \Gamma^* \).

We shall see (§11), that at least for a certain class of curves \( \Gamma^* \) the surface \( x = x(u, v), y = (u, v), z = z(u, v) \) is uniquely determined by the above conditions; therefore it is not advisable to restrict the problem by any further conditions.

2. The solution, given by Garnier, of the problem of Plateau, can be greatly simplified by the following

Theorem of Approximation.—Let \( \Gamma_n^* \) be a sequence of simple closed curves in the xyz-space with the following properties.

\( \alpha \) The length \( l(\Gamma_n^*) \) of \( \Gamma_n^* \) is uniformly bounded: \( l(\Gamma_n^*) \leq L \).

\( \beta \) The problem of Plateau (as stated above) is solvable for \( \Gamma_n^* \) (n = 1, 2, ...).

\( \gamma \) The sequence \( \Gamma_n^* \) converges in the sense of Fréchet toward a simple closed curve \( \Gamma^* \).