page 759, and volume 16 (1930), page 400. The theorem noted at the close of the second paragraph seems to be especially useful in dealing with these groups and it should supersede the corresponding theorems relating thereto, published in the articles just cited.

A necessary and sufficient condition that a group is decomposable into two cyclic subgroups is that it contains at least two cyclic subgroups such that the product of their orders divided by the order of their cross-cut is equal to the order of the group. In closing it may be desirable to direct attention to a very elementary infinite system of groups composed of groups which are separately decomposable into two non-invariant cyclic subgroups but not also into two cyclic subgroups of which at least one is invariant. Such a system may be constructed by forming the direct products in succession of two non-abelian groups of orders $pq$ and $pr$, respectively, where $p$, $q$, $r$ are prime numbers and $q$ may or may not be equal to $r$. All such groups are decomposable into two cyclic groups of orders $pq$ and $pr$, respectively, and the smallest group which belongs to this system is the holomorph of the symmetric group of order 6. In this case $p = 3$ and $q = r = 2$.


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**ON THE DUALITY THEOREMS FOR THE BETTI NUMBERS OF TOPOLOGICAL MANIFOLDS**

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The terms and notations used throughout this note will be those of Lefschetz.1 A topological manifold $M_n$ is a compact metric space whose defining neighborhoods are $n$-cells. $M_n$ can be covered by a finite set of $n$-cells. Any such set is called a covering set.

The Betti numbers of $M_n$ have been defined by W. Flexner2 by means of a nexus of oriented cells, called elemental cells, on $M_n$, which overlap each other. When these cells are small enough the Betti numbers of $M_n$ (absolute, mod $m$, etc.) can be calculated from the incidence relations of these cells as in the purely combinatorial case. The object of the present note is to announce, and briefly to describe, the proof of the duality theorems for these Betti numbers.

Because more than two elemental $n$-cells of a covering are in general incident with each elemental $(n - 1)$-cell of $M_n$, the construction of a dual complex seemed impossible, hence the relations of the $p$-cycles and
(n - p)-cycles depending on the manifold as a whole had to be established in some other way. W. Flexner proved that it was possible to define the Kronecker index \((\gamma^i_{n-p} \cdot \gamma^j_p)\) of cycles \(\gamma^i_{n-p}, \gamma^j_p\) on \(M_n\) in such a way that if either \(\gamma^i_{n-p}\) or \(\gamma^j_p \approx 0\), \((\gamma^i_{n-p} \cdot \gamma^j_p) = 0\). Since \(M_n\) has the Urysohn-Menger dimensionality \(n\), it has a homeomorph on an \(S_r\) as proved by Menger.\(^3\) Alexander has given an elementary proof for the special case of \(M_n\), which has been embodied in the paper by Flexner already quoted.\(^2\)

We identify \(M_n\) with its image in \(S_r\), which we call \(M_n\) from now on. Taking advantage of the immersion property we show that given a set of \(h\) independent cycles \(\gamma^i_p\), there exist \(h\) cycles \(\gamma^i_{n-p}\) such that

\[
(\gamma^i_{n-p} \cdot \gamma^j_p) = \delta_{ij}. \tag{1}
\]

By Alexander's duality theorem\(^4\) as extended by Alexandroff,\(^5\) to the set of non-bounding \(p\)-cycles \(\gamma^j_p\) on \(M_n\) there corresponds a set of \((r - p)\)-chains, \(C^j_{r-p}\), in \(S_r\), whose boundaries do not cut \(M_n\) and such that \((C^j_{r-p} \cdot \gamma^j_p) = \delta_{ij}\).

The intersection of \(C^j_{r-p}\) and \(M_n\) can be defined as an \((n - p)\)-cycle \(G^j_{n-p}\) in \(S_r\) and arbitrarily close to \(M_n\). Then \(G^j_{n-p}\) can be deformed into \(\gamma^i_{n-p}\) on \(M_n\), in such a way that

\[
(\gamma^i_{n-p} \cdot \gamma^j_p) = (C^j_{r-p} \cdot \gamma^j_p)_S = \delta_{ij}. \tag{2}
\]

From this follows that the cycles \(\gamma^i_{n-p}\) are independent, which leads immediately to the duality theorem.

Having given a brief outline of the proof, we will explain some of the steps in more detail. To obtain the intersection of \(M_n\) and \(C^j_{r-p}\) it is necessary to replace \(C^j_{r-p}\) by another chain \(D^j_{r-p}\) homologous to \(C^j_{r-p}\) in \(S-M\), and at an arbitrarily small distance from it. \(C^j_{r-p}\) is such that no cells of \(C^j_{r-p}\) of dimensionality < \(r - n\) meet \(M_n\). That the construction of \(D^j_{r-p}\) is possible follows from the duality theorem of J. W. Alexander.\(^4\) \(D^j_{r-p}\) also is made of cells so small that the intersection of each cell of \(D^j_{r-p}\) with \(M_n\) is entirely contained in an \(n\)-cell on \(M_n\). Lefschetz has shown (loc. cit., ch. 4) that the intersection of two arbitrary chains, each of whose boundaries does not meet the other chain, can be approximated by taking the intersection of simplicial chains approximating the arbitrary ones. If the approximations are close enough, all intersections due to various approximations are homologous in a neighborhood of the geometric or point-set intersection.

The intersection \(G^j_{n-p}\) of \(M_n\) and \(D^j_{r-p}\) is built up by taking first the intersection of the \((r - n)\)-cells of \(D^j_{r-p}\) with one of the \(n\)-cells \(E_n\) of \(M_n\) covering their geometric intersection. Then the intersection of the \((r - n + 1)\)-cells of \(D^j_{r-p}\) with the \(E_n\) covering their geometric intersections with \(M_n\) are each taken and modified so that the boundary of
the intersection is the intersection of the boundary with $M_n$, which was obtained when the $(r - n)$-cells were treated. This process is continued until the $(r - p)$-cells have been treated. Because of the modification at each step to secure a proper boundary for the intersection, the intersection of $M_n$ and $D^j_{r-p}$, defined as the sum of the intersections of its component cells, is a cycle.

Another device is used to prove that

$$(\gamma_{n-p}^j \cdot \gamma_p^j)_M = (D^j_{r-p} \cdot \gamma_p^j)_S.$$  \hspace{1cm} (3)

It amounts to choosing as the $h$ independent non-bounding cycles on $M_n$ a set with special properties. In the first place these particular cycles are composed of pieces each simplicial in some $n$-cell of $M_n$. The simplicial pieces are connected by point-sets on $M_n$, each of which is an arbitrarily close neighborhood of the boundary of a simplicial piece. This means that the non-simplicial parts of the cycles are neighborhoods of sets of dimensionality $p - 1$. The existence of such a cycle homologous to each arbitrary cycle is proved by means of the deformation theorem of Alexander and Veblen. Secondly, these cycles are altered and combined algebraically until they are such that the removal of $h$ of their simplicial $n$-cells will reduce to zero the $p$th connectivity of the subset of the points of $M_n$ carrying the independent cycles. This gives a way of forcing the intersection of $D^j_{r-p}$ and $\gamma_p^j$ into regions which can each be covered by an arbitrarily small $n$-cell of $M_n$.

From this point on there are no major difficulties. What happens essentially is that all required intersections are reduced to polyhedral intersections on an $E_n$ of $S_r$. In that region, for $r > 2n + 1$, a choice always possible, anyone covering $E_n$ of $M_n$ can be indefinitely approximated by a non-singular polyhedral cell which takes the place of $E_n$, and the intersection theory developed by Lefschetz leads to (3), then to (2) completing the proof of the duality relations.

Once the duality relations are proved we can apply the results obtained by Lefschetz for the coincidences and fixed points of continuous transformations of combinatorial manifolds. In particular we have an immediate definition for Brouwer's degree as in the case of absolute manifolds. For, assuming $M_n$ connected, the duality theorem implies that there exists a fundamental $n$-cycle $\gamma_n$ on $M_n$, and the degree of $T$ is merely the coefficient $d$ in the homology $T \cdot \gamma_n \approx d\gamma_n$. See in this regard recent papers by Wilson and Hopf.

In concluding it may be said that the methods here outlined enable us to develop a systematic intersection theory for topological manifolds immersed in one another.

1 Lefschetz. Topology, The Providence Colloquium Lectures. (Due to appear in the Fall of 1930.)
Belling suggested that the chromosome rings found in Oenothera by Cleland and others are to be explained as resulting from exchanges of ends between non-homologous chromosomes, so that one chromosome of a given complex is homologous at one end to one chromosome of a second complex, and at the other end to a different chromosome of the second complex. Håkansson and Darlington have elaborated this view. In a recent issue of this JOURNAL Cleland and Blakeslee have carried the analysis through in detail, showing that it gives self-consistent results. It enables one to predict the configurations of untried combinations, and is to a certain extent in agreement with the genetic data of Renner and Oehlkers.

We have studied cases in Drosophila that conform to the scheme that is required to fit Oenothera. The details of these experiments are now ready for publication, and will appear elsewhere. We wish here to point out their bearing on Oenothera problems, since it has been possible to carry out a far more detailed and accurate genetic analysis than will be possible in Oenothera for many years.

We have studied four cases of translocations involving the two large V-shaped pairs of autosomes (II and III) of Drosophila melanogaster. One case (translocation E) apparently arose spontaneously. A section from the end of the left limb of chromosome II became detached, and re-attached near the middle of the left limb of chromosome III. This case, while of interest in other connections, does not furnish a good parallel to Oenothera and need not concern us further here. The other three cases (translocations A, B and C) all arose in x-ray experiments, and are all reciprocal translocations—i.e., they represent an exchange of parts.