1. Introduction.—We present here the principal results of an investigation on the determination of lines of Julia of integral functions. Julia\(^4\) established the fact that each integral function \(f(z)\) possesses at least one ray (issuing from the origin) such that within every angle, however small, which the ray bisects, \(f(z)\) assumes infinitely often each value with the possible exception of at most one. Such a ray is commonly called a line of Julia, denoted by line \(J\). A number of writers have considered such half lines with regard to their properties, the degree of arbitrariness of the sets of such lines, and their location. We relate the existence of lines \(J\) to asymptotic rays, to the order of the function on and in the neighborhood of a given ray, to the type of the function on the ray, and to the zeros of the function. Montel’s\(^5\) theory of normal families is used throughout.

2. Sufficient Conditions.—We establish the following theorems.

**Theorem I.** If the integral function \(w = f(z)\) possesses two rays \(\theta = \theta_1\) and \(\theta = \theta_2\), \((z = r e^{i\theta}, 0 \leq \theta_1 < \theta_2 < 2\pi)\) such that

\[
\lim_{r \to \infty} f(re^{i\theta_1}) = a
\]

and

\[
\lim_{r \to \infty} f(re^{i\theta_2}) = b \neq \infty,
\]

where \(a \neq b\), then there exists at least one ray \(\theta = \phi, \theta_1 \leq \phi \leq \theta_2\) which is a line \(J\).

It may happen that a given function assumes each value except perhaps one infinitely often in some angle of opening \(\alpha\).\(^6\) It has not been proved in general\(^7\) that every such angle contains a line \(J\). Under each of two additional restrictions this fact can be established. We have

**Theorem II.** Let the function \(f(z)\) be analytic in an angle \(\theta_1 - \epsilon \leq \theta \leq \theta_2 + \epsilon\), \((0 < \theta_1 < \theta_2 < 2\pi, \epsilon > 0)\), and in the angle \(\theta_1 \leq \theta \leq \theta_2\) let it assume infinitely often every value with the possible exception of at most one. Furthermore, if either

I. \(\lim_{r \to \infty} f(re^{i\theta})\) exists for some \(\phi, \theta_1 < \phi < \theta_2\); or

II. \(f(z)\) is bounded on both \(\theta = \theta_1\) and \(\theta = \theta_2\), then there exists at least one line \(J\) in the given angle.
3. **Asymptotic Values.**—A ray \( \theta = \theta_0 \) on which a given integral function \( w = f(z) \) is such that either

\[
\lim_{r \to \infty} f(re^{i\theta}) = a,
\]

a finite, or for every positive \( M \) there exists a positive constant \( r(M) \) such that

\[
|f(re^{i\theta})| > M
\]

for \( r > r(M) \), is called an asymptotic ray of \( f(z) \).

It is well known that an asymptotic value of an integral function \( w = f(z) \) corresponds to a non-algebraic singularity of the inverse function \( z = \psi(w) \). Further, two finite asymptotic values \( a \) attained on different paths (1) and (2) are said to correspond to the same non-algebraic singularity of the inverse function if to each \( \epsilon > 0 \) there exists an \( r(\epsilon) > 0 \) such that for each pair of points \( z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2} \), where \( r_1 > r(\epsilon) \) and \( r_2 > r(\epsilon) \) there exists a path connecting \( z_1 \) and \( z_2 \), on which \( |f(z) - a| < \epsilon \).

If this is not the case, the asymptotic values are said to correspond to different non-algebraic singularities of the inverse function. The following theorems relate the existence of lines \( J \) to the non-algebraic singularities of the inverse function.

**Theorem III.** If the integral function \( w = f(z) \) possesses two rays \( \theta = \theta_1 \) and \( \theta = \theta_2 \), \( (0 \leq \theta_1 < \theta_2 < 2\pi) \), such that

\[
\lim_{r \to \infty} f(re^{i\theta}) = a \neq \infty
\]

and

\[
\lim_{r \to \infty} f(re^{i\theta}) = a,
\]

then either \( f(z) \) converges uniformly toward \( a \) in the whole angle \( \theta_1 \leq \theta \leq \theta_2 \), and the two values \( a \) correspond to the same non-algebraic singularity of the inverse function, or there exists at least one line \( J \) in the given angle.

The proof follows from an application of theorem I.9 The following partial converse is also true.

**Theorem IV.** Under the hypotheses of theorem III, if \( \theta = \phi_1, \theta_1 < \phi_1 < \theta_2 \), and \( \theta = \phi_2, \theta_2 < \phi_2 < \theta_1 + 2\pi \), are both lines \( J \), then the two values \( a \) attained asymptotically on \( \phi_1 \) and \( \phi_2 \) correspond to different non-algebraic singularities of the inverse function \( z = \psi(w) \).

By applying the preceding results one can obtain the following theorems.

**Theorem V.** A ray \( \theta = \theta_1 \), on which the integral function \( w = f(z) \) attains no limit, but such that in every angular neighborhood containing it there exist asymptotic rays, is a line \( J \).

**Theorem VI.** An integral function \( w = f(z) \) of finite order cannot converge on all rays to finite asymptotic values.
4. Order and Type.—We establish the following condition based on the order of a function in an angle.

**Theorem VII.** If the integral function \( w = f(z) \) possesses two rays \( \theta = \theta_1 \) and \( \theta = \theta_2 \) \((0 \leq \theta_1 < \theta_2 < 2\pi)\) such that

\[
|f(re^{i\theta_1})| = O(e^{\alpha})
\]

and

\[
|f(re^{i\theta_2})| = O(e^{\alpha - \eta}),
\]

where \( \eta \) is some positive constant, then there exists at least one line \( J \) on the boundary or in the interior of the angle \( \theta_1 \leq \theta \leq \theta_2 \).

This follows directly from an application of a recent theorem of Mandelbrojt.10

We introduce the notion of type on a ray in the following manner.

**Definition 1.**—Type.—An integral function \( w = f(z) \) \((z = re^{i\theta})\) of order \( \rho \) on a ray \( \theta = \theta_0 \) is said to be of type \( \mu(\theta_0) \) if

\[
\mu(\theta_0) = \lim_{r \to \infty} \log \frac{|f(re^{i\theta_0})|}{r^\rho}
\]

**Definition 2.**—Vanishing and Unbounded Types.—An integral function of order \( \rho \) on a sequence of rays \( \theta_n \) converging to \( \theta = \theta_0 \) is said to be of vanishing type on \( \theta = \theta_0 \) if

\[
\lim_{n \to \infty} \mu(\theta_n) = 0
\]

and of unbounded type if

\[
\lim_{n \to \infty} \mu(\theta_n) = \infty,
\]

where \( \mu(\theta_n) > 0 \) for all values of \( n \).

We find the following result on type.

**Theorem VIII.**—A ray \( \theta = \theta_0 \) on which the integral function \( w = f(z) \) is of finite order and vanishing or unbounded type is a line \( J \).

5. Zeros of an Integral Function.—It was shown by Gofficharoff11 that if \( \{a_n\} \) represents a sequence of zeros of an integral function clustering to a given ray, and such that \( \left| \frac{a_n + 1}{a_n} \right| < M \), where \( M \) is a positive constant independent of \( n \), and if the order of \( f(z) \) is \( \rho \leq 1/2 \), the ray is a line \( J \), while if the order of \( f(z) \) is \( \rho > 1/2 \), there is a line \( J \) within every angle of opening \( 2\pi - 2\pi/\rho \) containing the given ray. We obtain the following theorem on the zeros.

**Theorem IX.** Let \( w = f(z) \) be an integral function and let \( \{a_n\} \), \((n = 1, 2, \ldots)\) denote a sequence of zeros of \( f(z) \), such that \( n^\rho < |a_n| < n^\rho+1-\epsilon \), \( \rho > 0, \epsilon > 0 \), for all but a finite number of zeros; then, (I) if \( \{a_n\} \) be confined
to any angle with vertex at the origin, there is a line \(J\) in the given angle, or in it \(f(z)\) converges uniformly to zero; (II) if \(\{a_n\}\) clusters to a given ray, the ray is a line \(J\), or on it \(f(z)\) converges to zero.

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**ON THE TIME AVERAGE THEOREM IN DYNAMICS**

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1.—*Introductory.*—We consider a conservative dynamical system with an analytic Hamiltonian function \(H(p_1, \ldots, p_n, q_1, \ldots, q_n)\). The phases \(p_1, \ldots, p_n, q_1, \ldots, q_n\) of the system may be regarded as points \(P\) in the phase space. After elapsed of the time \(t\) a point \(P = P_0\) will be carried into a new position \(P_t = T_t(P)\). Apart from singularities, \(T_t(P)\) is well known to represent an analytic one parameter group of analytic one to one transformations of the phase space into itself,

\[T_s T_t = T_{t+s}, \quad (P_s)_t = P_{s+t}.\]

The phase volume is invariant under \(T_t\). As \(H = \text{const.}\) is an integral of the equations of motion, \(T_t\) implies a group on the \((2n - 1)\)-dimensional manifold \(H = c\). Similarly, we obtain a group on the \((2n - m - 1)\)-dimensional manifold \(H = c, H_s = c_s, v = 1, \ldots, m_1\) represent \(m\) further known integrals. In any case, a positive integral \(m = \int \rho \text{d}v, \rho > 0\), invariant under \(T_t\), is associated with the considered manifold. We suppose that the group is free of singularities, i.e., that \(T_t\) is analytically determined for all values of \(t\). This situation occurs in well-known cases, for instance, in the case of the regularized problem of three bodies.