ON THE DISTRIBUTIONS OF THE ZEROS OF CERTAIN
ANALYTIC FUNCTIONS

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In this note there are given some results of a study of the distribution of the zeros of a function of the form

\[ f(z) = \sum_{j=0}^{J} \exp (\lambda_j z^N + \ldots + \lambda_j z + \lambda_0), \]

(1)

where \( J \) and \( N \) are positive integers, and the \( \lambda \)'s are real or complex constants. It is assumed that we do not have \( \lambda_0 = \lambda_1 = \ldots = \lambda_J \), and that we do not have \( \lambda_0 = \lambda_1 = \ldots = \lambda_J \neq 0 \) for any \( n < N \). It is also assumed that no two terms in the right-hand member of (1) are identical or differ only by a constant factor.

The work reported here is essentially a generalization, in one direction, of certain well-known work by Wilder, Tamarkin, Pólya, Schwengler and others.\(^1\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\)

We first arrange \( f(z) \) in a certain normal form. The function can be written

\[ f(z) = \sum_{m=0}^{M} f_m(z) \exp \left[ g_m(z) + \mu_m z^N \right], \]

where: \( M \) is a positive integer; \( \mu_0, \ldots, \mu_M \), are distinct constants; \( g_m(z) \) is an integral rational function of degree less than \( N \); and \( f_m(z) \) is either the constant 1 or a function such as \( f(z) \) in which \( N \) is replaced by a positive integer \( N_m < N \). If \( f_m(z) \) is not constant, we can write

\[ f_m(z) = \sum_{n=0}^{M_m} f_{mn}(z) \exp \left[ g_{mn}(z) + \mu_{mn} z^{N_m} \right], \]

with stipulations similar to the above. And so on. We continue this process of arranging \( f(z) \) until it automatically terminates after a finite number of steps.

Let the points \( \mu_0, \ldots, \mu_M \), be plotted in the complex plane, and let the smallest convex polygon that contains these points in its interior or on its boundary be drawn. We call this polygon the primary critical polygon for \( f(z) \), and we denote it by the symbol \( P \). Let the subscripts be assigned so that \( \mu_0, \mu_1, \ldots, \mu_M \), are the vertices of \( P \) in counter-clockwise order, an arbitrarily chosen vertex being denoted by \( \mu_0 \). Let the side of \( P \) that follows the typical vertex \( \mu_\alpha \) in counter-clockwise order be denoted by \( L_\alpha \). From any point on \( L_\alpha \) draw the normal exterior to \( P \). Let \( \varphi_\alpha(0) \leq \varphi_\alpha \)
< 2π) be the angle between the positive real axis and this outward drawn normal. Let the vertex denoted by \( \mu_0 \) be selected so that \( \varphi_0 < \varphi_1 < \ldots < \varphi_{M'} \). From the origin draw the \((M' + 1)N\) rays \( R^{(\beta)}_a \) having the equations

\[
R^{(\beta)}_a : \text{amp} \ z = -\frac{\varphi_a}{N} - \frac{2\beta\pi}{N}, (\alpha = 0 \ldots M'), (\beta = 0 \ldots N - 1).
\]

We call these rays the primary critical rays for \( f(z) \). These rays divide the plane into \((M' + 1)N\) sectors, each of which is bounded by two of the rays and has none of the rays in its interior. We denote the one of these sectors which is bounded on the clockwise side by the ray \( R^{(\beta)}_a \) by the symbol \( S^{(\beta)}_a \). We consider \( S^{(\beta)}_a \) as not including the points on its bounding rays.

Let \( \alpha \) be any one of the integers \( 0 \ldots M' \). If \( f_a(z) \) is not constant, we construct, in the same way, its primary critical polygon, \( P_a \), its primary critical rays

\[
R^{(\gamma)}_{\alpha\beta}, (\beta = 0 \ldots M'_a), (\gamma = 0 \ldots N_a - 1),
\]

and the associated sectors \( S^{(\gamma)}_{\alpha\beta} \). Similarly, if \( \alpha \) is one of the integers \( 0 \ldots M' \), and if \( \beta \) is one of the integers \( 0 \ldots M'_a \), and if \( f_{\alpha\beta}(z) \) is not constant, we construct the primary critical polygon for \( f_{\alpha\beta}(z) \), the primary critical rays

\[
R^{(\delta)}_{\alpha\beta\gamma}, (\gamma = 0 \ldots M'_a), (\delta = 0 \ldots N_{\alpha\beta} - 1),
\]

and the associated sectors \( S^{(\delta)}_{\alpha\beta\gamma} \). And so on. We continue this process of constructing polygons, rays and sectors, until it automatically terminates after a finite number of steps.

We have defined the primary critical rays of \( f(z) \); now we proceed to define critical rays "of higher order." A secondary critical ray of \( f(z) \) is a primary critical ray of a function \( f_a(z) \), \( (\alpha = 0 \ldots M') \), which lies in one of the sectors \( S^{(\gamma)}_{\alpha} \ldots S^{(N - 1)}_{\alpha} \). A tertiary critical ray of \( f(z) \) is a secondary critical ray of a function \( f_a(z) \), \( (\alpha = 0 \ldots M') \), which lies in one of the sectors \( S^{(\gamma)}_{\alpha} \ldots S^{(N - 1)}_{\alpha} \). And so on. Note that in these definitions we have the same value of \( \alpha \) in \( f_a(z) \) and \( S^{(\beta)}_a \).

We are now ready to announce the chief results of the investigation in the form of the following three theorems. The proofs require nothing more than the use of familiar analytical processes, but they are long and complicated. They will be given later in another paper.

**Theorem 1.** There exists a set of half-strips, equal in number to the critical rays of \( f(z) \), each extending in the direction of a different one of the critical rays, such that each zero of \( f(z) \) is a point of one or more of these half-strips.

By a half-strip is meant the open set of points between two parallel
straight lines and on one side of a line perpendicular to these. It is natural to classify the half-strips referred to in Theorem 1 as primary, secondary, etc., according to the natures of the critical rays to which they correspond.

**Theorem 2.** The number of zeros of \( f(z) \) in a primary half-strip and within the circle \(|z| = r\) is equal, for \( r \) large, to

\[
\frac{l^N}{2\pi} \left[ 1 + O(l/r) \right],
\]

where \( l \) is the length of the side of \( P \) which corresponds to the half-strip in question.

**Theorem 3.** The number of zeros of \( f(z) \) within the circle \(|z| = r\) and in a non-primary half-strip corresponding to a critical ray of \( f(z) \) that is a critical ray of a function \( f_\alpha(z) \), \((\alpha = 0 \ldots M')\), is asymptotically equal, for \( r \) large, to the number of zeros of \( f_\alpha(z) \) in the same region.


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**ON THE BEHAVIOR OF THE \( n \)th ITERATE OF A FREDHOLM KERNEL AS \( n \) BECOMES INFINITE**

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Recent researches concerning geometrical probabilities which we will call "linked" (French: *en chaîne*) have demonstrated the importance of the study of the asymptotic behavior of the \( n \)th iterated kernels which present themselves in this problem.

We shall define iteration by the formula

\[
K^{(n+1)}(M,P) = \int_V K^{(n)}(M,Q)K^{(Q,P)}dQ;
\]

the kernels to be studied in this problem are those for which

(P) \[ K(M,P) \geq 0 \]

(T) \[ \int_V K(M,P)dP = 1. \]

A summary of the results obtained in this case has been published recently in the C. R. Acad. Sci., Paris.