we assume that every act of absorption from lower level $A$ to upper level $B$ is followed by an act of emission from $B$ to $A$. In other words, the number of atoms in state $B$ is defined by the number in $A$, and by the transition probability $A ightarrow B$. This is in contradiction to the assumption of thermodynamic equilibrium. Since in thermodynamic equilibrium the number of atoms in any state is fixed, it seems permissible to interpret our observations as indicating that the reversing layers of stars are not in thermodynamic equilibrium.

4 This tendency, if substantiated, would constitute an important argument against the hypothesis that the broadening of stellar absorption lines is caused entirely by axial rotation of the stars. Intensity measurements of a $Si$ III triplet in several "n" and "s" stars of class B revealed no differences in the gradient. (Struve and Elvey, *Astrophys. J.*, 72, 267 (1930).)
6 *Zeits. Astrophysik*, 1, 300 (1930).
8 Loc. cit., Figure 1.
9 In order to obtain $E \propto NH$, the lines must be very shallow in the center.

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**NOTE ON THE DIVISORS OF THE NUMERATORS OF BERNOULLI'S NUMBERS**

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We shall consider the Bernoulli numbers as rational fractions in their lowest terms with $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, etc. If $l$ is a prime $>3$, consider the set

$$B_1, B_2, \ldots, B_{l-3}.$$  \hspace{1cm} (1)

If $l$ does not divide the numerators of any of these Bernoulli numbers then $l$ is said to be regular, otherwise, irregular. Also if $B_1$ has a numerator divisible by a prime $p$ and $p$ is prime to $i$, then $p$ is said to be a proper divisor of the numerator; otherwise it is an improper divisor. In this connection we note the von Staudt-Clausen theorem$^1$ which states that

$$(-1)^n B_n = G_n + \frac{1}{2} + \frac{1}{\alpha} + \frac{1}{\beta} + \ldots + \frac{1}{\lambda}$$

---

in which \( \alpha, \beta \ldots \lambda \) are the numbers obtained by taking the divisors \( a, b \ldots l \) of \( n \), setting down the expression \( 2a + 1, 2b + 1 \ldots 2l + 1 \), and selecting only those integers in the latter set which are primes, and \( G_n \) is an integer. Hence the denominator of each Bernoulli number is determined. More generally if we set

\[
2n = 2^i p^a q^b \ldots r^s s^t \ldots
\]

where \( p, q \ldots r, s \ldots \) are all primes then the divisors of \( 2n \) divide into two classes; one class consisting of \( 2, p, q \ldots \) which appear as divisors in the denominator of \( B_n \) and the other class \( r^b, s^t \ldots \) appearing in the numerator of \( B_n \) as improper divisors.

We also have the congruence

\[
\frac{B_n}{n} \equiv (-1)^{\mu} \frac{B_{n + \mu}}{n + k\mu} \pmod{l}; \quad \mu = \frac{l - 1}{2}
\]

for any prime \( l \) with \( n \) an integer not divisible by \( (l - 1) \). We note also that \( B_{2\mu} \), where \( s \) is a positive integer, has, by the von Staudt-Clausen theorem, \( l \) as a divisor of the denominator. Hence, if \( l \) is a regular prime, the relation (2) shows that the integer \( l \) never appears as a proper divisor of the numerator of any Bernoulli number. On the other hand if \( l \) is irregular (2) shows that it is a proper divisor of the numerators of an infinity of Bernoulli numbers.

We have the relation

\[
B'_{n + \mu} - 2B'_{n + \mu} + B'_{n} \equiv 0 \pmod{l^2}
\]

where

\[
B'_{i} = \frac{(-1)^i B_i}{i}.
\]

From this we obtain

\[
B'_{n + \gamma\mu} \equiv yB'_{n + \mu} - (y - 1)B'_{n} \pmod{l^2}.
\]

If we now select \( l \) and \( n \) so that \( B'_{n} \equiv 0 \pmod{l} \) with \( B'_{n + \mu} \neq B'_{n} \pmod{l} \), it follows that there exists an integer \( y \) which yields \( B'_{n + \gamma\mu} \equiv 0 \pmod{l^2} \).

Pollaczek\(^6\) took \( l = 37, n = 16 \), which gave \( y = 7 \); hence \( B'_{143} \equiv 0 \pmod{37^2} \). He also found two other examples of this type. Hence the numerator of a Bernoulli number may be divisible by the square of a proper divisor.

Concerning the regular primes, the work of my assistants in recent years has shown that all primes \( < 307 \) are regular except

\[
37, 59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283, 293. \quad (3)
\]

In this connection Jensen\(^7\) proved that there is an infinity of irregular primes of the form \((4k + 3)\).

We may then state the
THEOREM: A regular prime never appears as a proper divisor of the numerator of any Bernoulli number. An irregular prime appears as a proper divisor of the numerators of an infinity of Bernoulli numbers. There is an infinity of irregular primes in the form of \((4k + 3)\). Contrary to the situation in connection with the divisors of the denominators of the Bernoulli numbers, a numerator of a Bernoulli number may be divisible by the square of a proper divisor. None of the primes \(< 307\) ever appear as proper divisors of the numerators of any Bernoulli numbers except those in the set \((3)\).

We now consider the relation of Bernoulli's numbers to the first factor of the class number of a cyclotomic field \(k(\zeta)\), where \(\zeta = e^{2\pi i/\ell}\). This first factor is generally expressed in the form

\[
h = \frac{f(Z)f(Z^2) \ldots f(Z^{\ell-2})}{(2i)^{\ell(\ell-3)/4}},
\]

where \(f(x) = r_0 + r_1x + r_2x^2 + \ldots + r_{\ell-2}x^{\ell-2}\), \(Z = e^{2\pi i/(\ell-1)}\), \(\ell\) is a primitive root of \(\ell\) and \(r_i\) is the least positive residue of \(r_i\), modulo \(\ell\). The writer has shown\(^8\) from this that

\[
h \equiv \frac{\prod (-1)^{(\ell a - 1)/2} B_{(\ell a + 1)/2}}{2^{\ell(\ell-3)/4}} (\text{mod } \ell^a)
\]

(4)

This congruence shows that each can be divisible by \(\ell^a\), with \(a\) as large as we please provided certain Bernoulli numbers have numerators divisible by sufficiently high powers of \(\ell\). On the other hand, the integer \(h\) is fixed in value, hence cannot be divisible by an arbitrarily high power of \(\ell\). It then follows that we have a limitation on the divisibility of the numerators of certain Bernoulli numbers by powers of \(\ell\). In particular, it may be shown, as in another paper,\(^9\) that if \(k\) of the Bernoulli numbers in the set \((1)\) have numerators divisible by \(\ell\), then \(h \equiv 0 (\text{mod } \ell^k)\). Hence, in particular, the value of \(h\) possibly limits the value of \(k\). Kummer\(^10\) stated without proof that the asymptotic value of \(h\) is

\[
\frac{\ell^{(\ell - 3)/4}}{2^{(\ell - 3)/2} \pi^{(\ell - 1)/2}}
\]

If we should assume Kummer's statement true, this furnishes an obvious limitation on the number of Bernoulli numbers in the set \((1)\) which have numerators divisible by \(\ell\). Also, the ideas as just expressed, suggest a number of problems. For example, we may use any expression of the type (4) without reference to the class number in order to possibly limit the Bernoulli numbers with numerators divisible by \(\ell\). It might be feasible to select a much smaller integer than \(h\), which would be the norm of an integer in the field \(k(Z)\) and which reduced \((\text{mod } \ell^a)\) to an expression involving Bernoulli numbers.
In order to secure a correct idea of the nature of the group concept it is desirable to consider not only group operations but also some of the closely related non-group operations. One of the most important instances of the latter type of operations from a historical point of view is multiplication by 0. Nicomachus (about 100 A.D.) remarked that 0 added to 0 gives 0, and the Hindus considered the four fundamental operations with respect to 0. If we combine by multiplication the totality of the rational numbers including 0 we do not obtain a group but if we omit 0 from this totality and combine the rest of the set by this operation there obviously results an abelian group of infinite order.

If we say that the distinct elements of a group must obey the associative law and assume that in group theory the operation of combining more than two elements at the same time is not considered, and if we postulate also that when any two of the three symbols in the equation $xy = z$ are replaced by group elements, which need not be distinct, then a third element of the same group is uniquely determined thereby, it results that the rational numbers including 0 obey the associative law but they do not obey the second of the two given group postulates. The following example of three distinct elements which obey the latter of the two given postulates without also obeying the former was given by J. Perott, *American Journal of Mathematics*, volume 11 (1889), page 101.

\[
\begin{align*}
1 \cdot 1 &= 1 & 1 \cdot 2 &= 3 & 1 \cdot 3 &= 2 \\
2 \cdot 1 &= 3 & 2 \cdot 2 &= 2 & 2 \cdot 3 &= 1 \\
3 \cdot 1 &= 2 & 3 \cdot 2 &= 1 & 3 \cdot 3 &= 3
\end{align*}
\]

It may be noted that these three elements are also commutative and