last upper and lower molars. *Chumashius balchi* cannot, therefore, be regarded as occurring in the direct line of development leading upward to the Recent tarsier.

In lower dental formula *Chumashius* exhibits a closer relationship to *Omomys* and *Hemiacodon* than to *Anaptomorphus* and *Tetonius*. The characters of the lower posterior premolars in *Uintanius* suffice to remove this Bridger genus from any close relationship with the Simi form. In addition to the presence of a comparable number of lower teeth the structural details of the dentition, in so far as these are known, point also to a kinship between *Chumashius* and that division of the Anaptomorphidae including *Omomys* and *Hemiacodon*. The characters of the Simi genus suggest a development from those of an antecedent form like *Omomys* or *Hemiacodon*. Possibly *Euryacodon* is also situated close to the stem form from which *Chumashius* has evolved. No previously described tarsiid from the North American Eocene has been found in association with a fauna as advanced as that occurring with *Chumashius balchi* at Locality 150 in the Sespe deposits of the Simi Valley region, California.\(^5\)

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**NOTE ON POLYNOMIAL INTERPOLATION TO ANALYTIC FUNCTIONS**

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Let \( C \) be a Jordan curve of the \( z \)-plane, and let the function \( w = \phi(z) \) map the exterior of \( C \) onto the exterior of the unit circle \(|w| = 1\) so that the points at infinity correspond to each other. Let \( C_\rho \) denote generically the curve \(|\phi(z)| = \rho > 1\) exterior to \( C \).

If the function \( f(z) \) is single-valued and analytic interior to \( C_\rho \) but has a singularity on \( C_\rho \), then any sequence of polynomials \( p_n(z) \) of respective degrees \( n \) is said to converge to \( f(z) \) on \( C \) with the greatest geometric degree of convergence provided for every \( R < \rho \) there exists \( M \) dependent on \( R \) but not on \( n \) or \( z \) such that we have

\[
|f(z) - p_n(z)| \leq \frac{M}{R^n} \text{ on } C.
\]
Under the hypothesis made on $f(z)$, it is known that there exists no sequence $\phi_n(z)$ such that (1) is valid for $R > \rho$.

The concept of greatest geometric degree of convergence is of central importance in the study of polynomial approximation to analytic functions. Many well-known sequences of polynomials, such as those of best approximation in various senses, converge on the given point set with the greatest geometric degree of convergence. Under the hypothesis that the sequence $\phi_n(z)$ converges to $f(z)$ on $C$ with the greatest geometric degree of convergence, it is true that the sequence $\phi_n(z)$ converges to $f(z)$ at every point interior to $C$, uniformly on any closed point set interior to $C$, and converges uniformly in no region containing in its interior a point of $\mathcal{E}$.

The object of the present note is to prove the following:

Let the points $z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)}, z_{n+1}^{(n)}$, $n = 0, 1, 2, \ldots$, lie on or within $C$ or more generally have no limit point exterior to $C$. Let $\phi_n(z)$ denote the polynomial of degree $n$ found by interpolation to $f(z)$ in the points $z_i^{(n)}$. Then a necessary and sufficient condition that for every $f(z)$ analytic on and within $C$ the sequence $\phi_n(z)$ should converge to $f(z)$ on $C$ with the greatest geometric degree of convergence is

$$\lim_{n \to \infty} [(z - z_1^{(n)})(z - z_2^{(n)}) \ldots (z - z_{n+1}^{(n)})]^{1/(n+1)} = r|\phi(z)|,$$

(2)

for $z$ exterior to $C$, where $r$ is the Robin's constant (capacity or transfinite diameter) of $C$.

If the points $z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)}, z_{n+1}^{(n)}$ are not all distinct, the condition of interpolation naturally requires agreement of suitable derivatives of $\phi_n(z)$ and $f(z)$ at the multiple points.

We introduce the notation

$$\omega_n(z) = (z - z_1^{(n)})(z - z_2^{(n)}) \ldots (z - z_{n+1}^{(n)}).$$

A suitably chosen branch of the function $[\omega_n(z)]^{1/(n+1)}$ is analytic except at infinity exterior to any $C$, which contains no point $z_i^{(n)}$, and at infinity has the derivative unity with respect to $z$. For sufficiently large $n$, each factor $z - z_i^{(n)}$ is uniformly limited in any closed finite region of the plane, and hence the functions $[\omega_n(z)]^{1/(n+1)}$ are similarly limited and form a normal family in the region exterior to $C$ but not containing the point at infinity.

We are here using a slight extension of the usual notion of normal family, for the functions $[\omega_n(z)]^{1/(n+1)}$ are not necessarily analytic at each point exterior to $C$. Nevertheless, in any closed limited region exterior to $C$, at most a finite number of those functions fail to be analytic, so the entire set of functions has the characteristic properties of a normal family.

We proceed to prove the sufficiency of condition (2). The validity of condition (2) implies the uniformity of approach to the limit on an arbitrary closed limited point set exterior to $C$, since the functions $[\omega_n(z)]^{1/(n+1)}$
form a normal family. Let \( f(z) \) be analytic interior to \( C_p \) but have a singularity on \( C_s \) and let \( R < \rho \) be arbitrary. Choose \( R_1, R < R_3 < \rho \), and \( R_3, 1 < R_3 < R_1/R \).

For sufficiently large \( n \), the points \( z_n^{(n)} \) lie interior to \( C_{R_1} \), and we have the well-known formula

\[
 f(z) - p_n(z) = \frac{1}{2\pi i} \int_{C_{R_1}} \frac{\omega_n(z)f(t) dt}{\omega_n(t)(t - z)}, \quad z \text{ interior to } C_R. \tag{3}
\]

Thus we have by the use of an obvious inequality,

\[
 \lim_{n \to \infty} \max_{z \in C_{R_1}} |f(z) - p_n(z)|, \quad z \text{ on } C_{R_1}^{1/(n+1)} \leq \frac{R_3}{R_1} < \frac{1}{R'},
\]

for we have uniformly \( \lim \) \( \omega_n(z) \)^\((n+1) = \rho R_3 \), \( z \) on \( C_{R_1} \); \( \lim \) \( \omega_n(t) \)^\((n+1) = \rho R_1 \), \( t \) on \( C_{R_1} \). This inequality holds for \( z \) on \( C_{R_1} \) and hence in particular for \( z \) on \( C \); we have shown that the sequence \( p_n(z) \) approaches \( f(z) \) on \( C \) with the greatest geometric degree of convergence.

In the proof of the necessity of condition (2) it happens that we need merely study the expansion of the function \( f(z) = 1/(t - z) \), where \( t \) is suitably chosen exterior to \( C \). For this particular function we have

\[
 f(z) - p_n(z) = \frac{\omega_n(z)}{\omega_n(t)(t - z)};
\]

it may be verified directly that \( p_n(z) \) as thus defined is a polynomial in \( z \) of degree \( n \), and it is seen by inspection that \( f(z) \) and \( p_n(z) \) are equal in the points \( z = z_n^{(n)} \).

By hypothesis the sequence \( p_n(z) \) converges to \( f(z) \) on \( C \) with the greatest geometric degree of convergence. Then we have

\[
 \lim_{n \to \infty} \sqrt[n+1]{\frac{M_n}{\omega_n(t)}} \leq \frac{1}{R}, \quad M_n = \max_{z \in C} |\omega_n(z), \quad z \text{ on } C, \quad |\phi(t)| = R.
\]

But we have also

\[
 |\omega_n(t)| \leq M_n R^{n+1},
\]

whence

\[
 \sqrt[n+1]{\frac{M_n}{\omega_n(t)}} \leq \frac{1}{R}, \quad \lim_{n \to \infty} \sqrt[n+1]{\frac{M_n}{\omega_n(t)}} \leq \frac{1}{R}.
\]

It follows finally that we have

\[
 \lim_{n \to \infty} \sqrt[n+1]{\frac{M_n}{\omega_n(t)}} = \frac{1}{R'}
\]

If now \( t_1 \) is any other point on the curve \( C_{R_1} \), we likewise have
Each factor \(|z - z_k^{(n)}|\) is uniformly bounded from zero, for \(n\) sufficiently large, in any closed region exterior to \(C\). Hence no limit function of the normal family \([\omega_n(z)]^{1/(n+1)}\) can vanish exterior to \(C\). Any limit function of the family is of constant modulus on every \(C_R\), by (4).

Let us choose \(\phi(z)\) so that the function \(r\phi(z)\) has a Laurent development at infinity of the form \(z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\); this is possible by the definition of \(r\). Then the functions

\[
\frac{[\omega_n(z)]^{1/(n+1)}}{r \phi(z)}
\]

are all analytic at infinity (when suitably defined) and take on the value unity at infinity. The functions (5) also form a normal family in the region exterior to \(C\), including the point at infinity, and any limit function of the family is of constant modulus on every \(C_R\), is analytic and different from zero exterior to \(C\), and has the value unity at infinity. Hence, every such limit function is itself unity and (2) is established.

Condition (2), valid uniformly on any closed limited point set exterior to \(C\), implies that the points \(z_k^{(n)}\) have no limit point exterior to \(C\).

Our theorem can be extended to point sets \(C\) much more general than Jordan regions; it is sufficient if the complement \(K\) of \(C\) is connected, and regular in the sense that Green's function for \(K\) with pole at infinity exists.

If condition (2) is satisfied uniformly on any closed limited point set exterior to \(C\), it will be noted that the lemniscates

\[|\omega_n(z)| = r^{n+1}R^{n+1}\]

approximate uniformly the curve \(C_R\). The reciprocal is also true. Approximation not of curves \(C_R\) but of \(C\) itself by lemniscates was considered by Faber\(^4\) in connection with the condition

\[\lim_{n \to \infty} p_n(z) = f(z), \text{ uniformly on } C,\]

for an arbitrary \(f(z)\) analytic on and within \(C\), but Faber's conclusions require essential modification.

It frequently arises in practice that the polynomials \(\omega_n(z)\) are not given directly, but rather polynomials \(\psi_n(z)\) which are constant multiples of the \(\omega_n(z)\); the coefficient of \(z^{n+1}\) is not necessarily unity. Formula (3) is clearly valid if the \(\omega_n(z)\) and \(\omega_n(t)\) are replaced by \(\psi_n(z)\) and \(\psi_n(t)\). The condition
uniformly on any closed limited point set exterior to \( C \), where \( \sigma \) is a non-vanishing constant, is also sufficient that (1) should be valid for every \((z)\) analytic on and within \( C \), as follows by the reasoning already used.

It is worth noticing that the validity of (6) \( (\sigma \neq 0) \) uniformly on any closed limited point set exterior to \( C \) implies the corresponding condition (6) where \( \psi_n(z) \) is replaced by its derivative \( \psi'_n(z) \). This new condition implies that the roots of the \( \psi_n(z) \) have no limit point exterior to \( C \), so the property expressed in our theorem, if satisfied for interpolation in the roots of \( \psi_n(z) \), is also satisfied for interpolation in the roots of \( \psi'_n(z) \), \( \psi''_n(z) \), etc.

There are a large number of illustrations in the literature \(^4\) of the use of the condition of the theorem (taken together with the auxiliary remarks) to prove convergence to \( f(z) \) of the sequence \( p_n(z) \), ordinarily with less precise results on convergence than we have established here: interpolation in the roots of unity and in certain real points studied by Runge, generalization of interpolation in the roots of unity studied by Fejér and Kalmár, interpolation in the roots of Faber’s polynomials and the roots of Tchebycheff polynomials studied by Faber, interpolation in the roots of Szegö’s polynomials studied by him, and interpolation in points introduced by Fekete. For the case that \( C \) is a line segment or the unit circle, the points \( z_k^{(n)} \) may also be chosen as the roots of various orthogonal polynomials. \(^4\)

Our main theorem is clearly related to the following theorem, transmitted to the writer some time ago by Mr. Fekete; the proof is still unpublished:

Let \( C \) be an arbitrary Jordan curve, and let the points \( z_1^{(n)} \), \( z_2^{(n)} \), \ldots, \( z_{n+1}^{(n)} \) lie on or within \( C \). Then a necessary and sufficient condition that the sequence \( p_n(z) \) should converge to \( f(z) \) interior to \( C \) (not necessarily uniformly) for every \( f(z) \) analytic on and within \( C \), is

\[
\lim_{n \to \infty} \sqrt[n+1]{M_n} = r, \quad M_n = \max \{ |\omega_n(z)|, \ z \text{ on } C \}.
\]

If this condition is satisfied, we have \( \lim_{n \to \infty} p_n(z) = f(z) \) uniformly on any closed set interior to \( C \), for every such \( f(z) \).

\(^1\) See the writer’s forthcoming Approximation of Polynomials in the Complex Domain, Mémorial des sciences mathématiques.

\(^3\) Loc. cit. §9.

\(^5\) For detailed references, the reader may refer to Walsh, loc. cit., Ch. V.