\[ \gamma_r = \alpha_r + \beta_r \pmod{2} \quad (r = -n, -n + 1, \ldots, 0, 1, 2, \ldots, n, \ldots). \]

8. Our other generalization is that in which \( G^1 \) instead of being too large is too small. Suppose, for instance, that \( G^1 \) denotes Hilbert space \( \sum x_n^2 < \infty \), and that the product of \( \{x_n\} \) and \( \{y_n\} \) is \( \{x_n + y_n\} \). The most general character which is continuous is

\[ \exp \left\{ 2\pi i \left( \sum \lambda_n x_n \right) \right\}, \]

where \( \sum \lambda_n^2 \) must converge. And, in this case, \( G \) also will be a Hilbert space. We use a very valuable theorem due to Banach\(^4\) to show that if \( G^1 \) denotes a separable metric space of type\(^5\) \( B \), and if the fundamental operation is that of addition, then we can find a denumerable set of characters of \( G^1 \) such that, at any rate formally, every continuous character can be expressed as a finite or infinite product of powers of these fundamental characters.

\(^1\) International Research Fellow.
\(^4\) S. Banach, Théorie des opérations linéaires, Warszawa, 1932.
\(^5\) See, e.g., Banach, loc. cit., Chapter V.

**GEOMETRY OF THE HEAT EQUATION: SECOND PAPER**

**THE THREE DEGENERATE TYPES OF LAPLACE, POISSON AND HELMHOLTZ**

**By Edward Kasner**

**Department of Mathematics, Columbia University**

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1. The classical problem of the flow of heat by conduction in the plane suggests a corresponding problem in the differential geometry of families of plane curves. With a given flow of heat we associate a family of *heat curves*: namely, the complete set of \( \infty \) isothermals throughout the flow; one curve for each temperature, for each value of the time. Then the physics of the flow is, at least in part, reflected in the geometry of the family of curves. We studied some simple questions of this geometry in our first paper, especially rectilinear and circular solutions.

Analytically, a flow of heat is described by giving the temperature \( \nu \) as a function of position and time

\[ \nu = \varphi(x, y, t), \]

where the function \( \varphi \) must be a solution of Fourier's equation for the conduction of heat
\[ \varphi_{xx} + \varphi_{yy} = \varphi, \quad \text{or} \quad \Delta \varphi = \varphi. \]  
(2)

(We may assume that the coefficient of \( \varphi \) is unity by changing the unit of time, if necessary.) Then (1) is also the equation of the corresponding family of heat curves, if we think of \( \nu \) and \( t \) as parameters.

Heat families, then, are a class of two-parameter families of plane curves. In general they therefore contain \( \infty^2 \) distinct curves. But in certain cases the two parameters may combine so as to amount to just a single essential parameter. Then there are only \( \infty^1 \) curves, and the heat family is degenerate. Instead of varying with the time, the same \( \infty^1 \) curves serve as the lines of constant temperature all through the flow (though of course the temperature attached to a particular curve may change with the time).

For instance, degeneracy must occur when \( t \) is actually absent from \( \varphi \), that is, in the case of steady flow. In fact (1) is then an ordinary isothermal (or Laplacian) family. But there are less obvious types, as stated below in Theorem II.

Clearly, if we wish to give a geometrical classification of families of heat curves, a first step must be to distinguish these special sets of \( \infty^1 \) curves from the general case. We therefore have the problem: to determine all degenerate families of heat curves.

Our main result is that there are three distinct types of degeneration of the Fourier heat equation. These may be connected with the equations of Laplace, Poisson and Helmholtz.

We first prove the following:

**Theorem I:** A one-parameter system of curves can be identified with a degenerate family of heat curves if and only if its equation can be put in the form

\[ g(x, y) = \text{const.}, \]  
(3)

where \( g \) satisfies an equation of the form

\[ \Delta g = ag + b, \]  
(4)

where \( a \) and \( b \) are constants, and \( \Delta \) denotes the Laplacian operator.

From this we deduce easily certain normal forms for all degenerate families:

**Theorem II:** Any degenerate family (by means of a similitude transformation) can be given the equation \( g(x, y) = \text{const.} \), where \( g \) is a solution of either \( \Delta g = 0 \), or \( \Delta g = 1 \), or \( \Delta g = \pm g \).

**Theorem III:** A family under one of these types cannot be under either of the others except that all three types include parallel straight lines and concentric circles (equations (11) and (12) below).

*There are three distinct types of degenerate heat families:*

**Type I:** defined by the Laplace equation

\[ \Delta g = 0. \]
TYPE II: defined by the special Poisson equation
\[ \Delta g = 1. \]

TYPE III: defined by the Helmholtz-Pockels equation
\[ \Delta g = \pm g. \]

In the real domain the third type divides into two distinct cases,

\[ \text{III}_1 : \Delta g = +g, \quad \text{and} \quad \text{III}_2 : \Delta g = -g. \]

We have then more accurately four types in all, namely, I, II, III, and III,.

It is obvious that the equation \( \Delta g = -1 \) is equivalent in the real domain to \( \Delta g = +1 \); hence type II requires no subdivision. On the other hand to reduce \( \Delta g = -g \) to \( \Delta g = +g \) one would have to replace \( x, y \) by \( ix, iy \).

2. Proofs.—Assuming that \( \varphi \) is analytic in \( t \), we can find the general solution of (2) explicitly in the form
\[ \varphi = f + tA f + \frac{t^2}{2!} A^2 f + \ldots, \tag{5} \]
where \( f(x, y) \) is an arbitrary function.

The condition for (1) to represent only \( \infty^1 \) curves can be stated in this way: \( x \) and \( y \) must be separable from \( t \) within \( \varphi \); that is,
\[ \varphi(x, y, t) = \text{a function of } u \text{ and } t, \tag{6} \]
where \( u \) is some function of \( x \) and \( y \).

It follows that each coefficient of (5) must separately be a function of \( u \) alone. It is no restriction to take \( f(x, y) \) as equal to \( u \). Hence \( f(x, y) \) is to be determined by the infinite set of conditions
\[ \Delta f = F(f), \tag{7_1} \]
\[ \Delta^2 f = G(f), \tag{7_2} \]
\[ \ldots \]

The function \( \varphi \) will then be found by substituting \( f \) into (5). The heat curves will be the one-parameter system
\[ f(x, y) = \text{const.} \tag{8} \]

From (7_1) we obtain by differentiation
\[ \Delta^2 f = F'F + (f_x^2 + f_y^2)F'' \tag{9} \]
Supposing first that \( F'' \neq 0 \), we must have, according to (7_2),
\[ f_x^2 + f_y^2 = \text{a function of } f. \tag{10} \]
But this is exactly the condition for (8) to represent a family of parallel curves. Then from (7_1) alone we readily prove that these parallel curves are either circles or straight lines. The solutions must be therefore the
parallel lines and concentric circles whose equations we determined in our first paper:

\[ y = \beta(v, t), \quad \text{with} \quad \beta^2 z_1 - \beta_{rr} = 0, \]  
\[ x^2 + y^2 = \rho(v, t), \quad \text{with} \quad \rho^2 (1 + \rho_t) - \rho_{rr} = 0. \]

If \( F^v \equiv 0 \), we have from (71)

\[ \Delta f = af + b. \]  

(13)

Then (72), \ldots are all satisfied. Moreover, the \( f \)'s for parallel lines and concentric circles can also be included in (13). Theorem I is therefore proved.

For an \( f(x, y) \) obeying (13) it is easy to calculate from (5) the explicit form of \( \varphi \), at least as far as \( t \) is concerned. If \( a \neq 0 \), (1) becomes (to some temperature scale)

\[ v = (f + b/a)e^{at}; \]  

(14a)

and if \( a = 0 \), we find

\[ v = f + bt. \]  

(14b)

Theorem IV: Hence in all possible degenerate cases (except of course the parallel straight lines (11) and the concentric circles (12), where \( t \) may appear in complicated algebraic or transcendental form), the temperature \( v \) is either independent of the time \( t \), or else is linear in \( t \), or else contains \( t \) exponentially. This corresponds to our three types I, II, III.

The family (8) is not changed as a set of curves if we replace \( f \) by any function of itself. An \( f \) equivalent in this sense to any \( f \) under (13) will be found among the solutions of either \( \Delta f = 0 \), or \( \Delta f = 1 \), or \( \Delta f = af \). If we allow magnification of the \( xy \) plane, the third equation can be specialized to \( \Delta f = \pm f \), and we thus have Theorem II.

Furthermore, for any set of curves in two of the three types, we quickly prove that \( f \) obeys (10). As before, this means either parallel lines or concentric circles. Both of these families can actually be described by \( f \)'s in all three classes, as Theorem III asserts, as follows.

Assuming our parallel straight lines horizontal we find that

\[ y = \text{const.} \quad \text{Type I} \]
\[ \frac{1}{2} y^2 = \text{const.} \quad \text{Type II} \]
\[ e^y = \text{const.,} \sin y = \text{const.} \quad \text{Types III}_1 \text{ and III}_2 \]

are the proper forms.

For concentric circles about the origin the results are

\[ \log r = \text{const.} \quad \text{Type I} \]
\[ \frac{1}{4} r^2 = \text{const.} \quad \text{Type II} \]
\[ I_0(r) = \text{const.,} \ J_0(r) = \text{const.} \quad \text{Types III}_1 \text{ and III}_2 \]
where $J_0$ is a standard Bessel function, and $I_0(r) = J_0(ir)$, and $r$ denotes the radius $(x^2 + y^2)^{1/2}$.

This completes the proof of Theorem III. Our three types of degeneration are therefore distinct in the sense that no family of $\infty^1$ curves, other than parallel straight lines and concentric circles, can belong to more than one type. Any non-trivial degenerate heat family belongs to one and only one of the types defined by the Laplace, Poisson and Helmholtz equations.

3. Characterizations.—We have found that all degenerate heat families can be classified under our three types I, II, III. This classification is analytic, depending on the form of the differential equations; but it can be translated, though not easily, into purely geometric form. The complete discussion is due to one of my students, Aaron Fialkow, who will publish a separate paper soon.\(^3\)

For the first type, where the Laplace equation is obeyed, the result is known. Consider the given $\infty^1$ curves and the orthogonal trajectories, forming an ordinary isothermal net. At each point the sum of the rates of change of the curvatures with respect to the arcs is zero. That is

$$\frac{d\gamma}{ds} + \frac{d\gamma_1}{ds_1} = 0$$

is the complete geometric equivalent of the Laplace equation

$$\varphi_{xx} + \varphi_{yy} = 0.$$

This property (15), since it involves first derivatives of curvatures, is geometrically of the third order. In can easily be restated in terms of osculating parabolas, or centers of curvature of evolutes. I have given another characterization, in terms of isogonal trajectories, in Math. Annalen, vol. 59, p. 352-354 (1904).

The fact that types II and III do not possess properties of third order, but do possess (individually) two properties of fourth order was established by the writer in his seminar of 1917. Fialkow finds these properties in explicit form and proves for the first time their sufficiency. Thus we have

**Theorem V:** While type I is characterized by one intrinsic geometric property of third order, namely (15), type II requires two (complicated) properties of fourth order, and the same is true of type III.

Fialkow has shown that the following geometric property (involving both tangential and normal intrinsic derivatives)

$$\gamma_1 \left( \frac{d^2 \gamma}{ds^2} - \frac{d^2 \gamma_1}{dn_1 ds_1} \right) + \left( \frac{d\gamma_1}{dn_1} - \gamma_1 \right) \left( \frac{d\gamma}{ds} + \frac{d\gamma_1}{ds_1} \right) = 0$$

characterizes the general class of equations $\Delta u = F(u)$, which includes our three types as particular species.
4. In a later paper I shall extend the main results of the first and second papers to three dimensions, that is, to families of heat surfaces. There are no solutions with \( \omega^2 \) planes (in the real domain). The three dimensional equations of Laplace, Poisson and Helmholtz again control the degenerate solutions.


2 Theorem I can be proved without assuming analyticity. Such proofs were given in my seminar in 1916, and again this year in shorter form by G. Comenetz. Hence all the results of the present paper are valid if we assume merely continuity and existence of partial derivatives.


THEORY OF ELASTIC SYSTEMS VIBRATING UNDER TRANSIENT IMPULSE WITH AN APPLICATION TO EARTHQUAKE-PROOF BUILDINGS

By M. BIOT

DEPARTMENT OF AERONAUTICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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Vibrating Systems under Transient Impulse.—The following theory gives a method of evaluating the action of very random impulses on vibrating systems (i.e., effect of static on radio-circuits or earthquakes on buildings). In the following text, we will use the language of mechanics.

Consider a one-dimensional continuous elastic system without damping. The free oscillations are given by the solutions of the homogeneous integral equation

\[ y = \omega^2 \int_a^b \rho(\xi) \alpha(x\xi)y(\xi)d\xi. \]

Due to the nature of the kernel there exists an infinite number of characteristic values \( \omega_i \) of \( \omega \) and of characteristic functions \( y_i \) solutions of this equation. These functions give the shape of the free oscillations of the system. They are orthogonal and have an arbitrary amplitude. This amplitude may be fixed by the condition of normalization,

\[ \int_a^b \rho(\xi)y_i^2(\xi)d\xi = 1. \]

We now suppose that certain external forces \( f(x) \) are acting on the system,