ON THE INVARIANCE OF GENERAL INTELLIGENCE

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Communicated July 5, 1933

I have discussed in a special case\footnote{This is a reference to a previous study or discussion.} the conditions under which Spearman's \( g \) remains invariant when the test scores are combined linearly. Irwin\footnote{This is a reference to another study or discussion.} has remarked that there would appear to be particular psychological importance to those transformations which conserve \( g \). It may be that some further comments will be useful.

Let the tests be \( a, b, c, \ldots \) with

\[
a = r_{ag}g + \sqrt{1 - r_{ag}^2} s_a = \cos \alpha g + \sin \alpha s_a \tag{1}
\]

and similar expressions for \( b, c, \ldots \). These \( k \) sets of \( n \) test scores may be represented as \( k \) unit vectors in a space of \( n \) dimensions. The \( k + 1 \) unit vectors \( g, s_a, s_b, \ldots \) being by hypothesis uncorrelated will be \( k + 1 \) mutually perpendicular unit vectors. If we are to have a set of \( k \) tests \( a', b', c', \ldots \) formed as linear combinations of \( a, b, c, \ldots \) with the same \( g \), the new specifics \( s'_a, s'_b, s'_c, \ldots \) must be linear combinations of \( s_a, s_b, s_c, \ldots \) as

\[
s'_a = l_a s_a + m_a s_b + n_a s_c + \ldots, \quad s'_b = l_b s_a + m_b s_b + n_b s_c + \ldots, \tag{2}
\]

etc., with such conditions as

\[
l^2_a + m^2_a + n^2_a + \ldots = 1, \quad l_i l_j + m_i m_j + n_i n_j + \ldots = 0 \tag{3}
\]

upon the \( k^2 \) constants \( l_i, m_i, n_i, \ldots \) to ensure that \( s'_a, s'_b, s'_c, \ldots \) form a new mutually orthogonal set of \( k \) unit vectors.\footnote{This is a reference to a previous study or discussion.} Substituting \( s_a = a \csc \alpha - g \cot \alpha \) from (1) with (2), introducing \( s'_a = a' \csc \alpha' - g \cot \alpha' \), and expressing the condition that \( g \) shall not enter explicitly into the relation between \( a, b, c, \ldots \) and \( a', b', c', \ldots \), we find as the equations of combination\footnote{This is a reference to a previous study or discussion.}

\[
a' \csc \alpha' = l_1 \csc \alpha a + m_1 \csc \beta b + n_1 \csc \gamma c + \ldots \tag{4}
\]

and

\[
cot \alpha' = l_1 \cot \alpha + m_1 \cot \beta + n_1 \cot \gamma + \ldots \tag{5}
\]

with similar expressions for \( b', c', \ldots \).

The \( k \) linear combinations (4) with the \( k \) conditions (5) are precisely those which leave \( g \) invariant. By virtue of (3) the constants \( l_i, m_i, n_i, \ldots \) are those of an orthogonal transformation. With the introduction in (4) of the multipliers \( \csc \alpha, \csc \beta, \ldots \) the transformation of \( a, b, c, \ldots \) into \( a', b', c', \ldots \) is not orthogonal but a sort of projection on the \( k \)-space of \( a, b, c, \ldots \) of an orthogonal transformation in the \( (k + 1) \)-space of \( g, s_a, s_b, \ldots \). The constants \( l_i, m_i, n_i, \ldots \) cannot all be positive or
zero, some must be negative, and thus the conclusion I reached in the simpler case is generalized: No replacement of \( a, b, c, \ldots \) by linear combinations of them can conserve \( g \) unless negative multipliers are permitted. Psychologists may object to subtracting scores even though they add them and subtraction but undoes addition; but at any event subtraction and addition either both come in or both stay out in considering possible linear combinations conserving \( g \). If both must stay out there can be no psychological interest in combinations which conserve \( g \), because there are none.\(^7\)

An important fact about the Spearman analysis into general and specific is that three tests are necessary and more than three are inconsistent unless the tetrad relations vanish. Such psychological tests as I have seen are formed of a large number of questions calling for responses of a more or less similar sort; the large number of questions being presumably necessary for reliability and the similarity for specializing the test to some aptitude or accomplishment. If the test be divided into three at random or if three equivalent tests be set up they will permit an analysis into general and specific factors. (By equivalent tests we should mean statistically those which had the same mean scores, the same dispersion of the scores about the mean and the same reliability.) The relation between the reliability of a test as measured by its correlation \( r_{aa} \) with itself and the reliability \( r_{tt} \) of the test made up of merging \( k \) equivalent tests is

\[
\frac{1}{r_{tt}} - 1 = \frac{1}{k} \left( \frac{1}{r_{aa}} - 1 \right).
\]

(6)

We are supposing that \( k = 3 \). The general factor \( g \) common to the three a's which may be called \( a, b, c \) will be found from

\[
r_{ag} = r_{bg} = r_{cg} = \sqrt{r_{aa}}, \quad Z = 3(1/r_{aa} - 1)^{-1} = (1/r_{tt} - 1)^{-1}
\]

\[
l = \frac{a + b + c}{3} \sqrt{r_{tt}}, \quad r_{tg} = r_{tt}, \quad r_{tg}^2 = 1 - r_{tt}.
\]

Thus the correlation of the total test or team with the general factor is the square root of the reliability of the test as measured by \( r_{tt} \). However, in this case we must consider the general factor as due both to the general intelligence and to the specific ability determined by the test and the undetermined part of this general factor would be due merely to that sort of chance fluctuation which produces the degree of unreliability which is present; the specifics would have no significant content.

In the analysis of a hierarchical system we introduce \( s_1, s_2, s_3, \ldots \) along with \( g \) in such a manner that the \( k(k + 1)/2 \) correlations of the \( k + 1 \) series each with each is zero. In case the s's were purely fortuitous as in the analysis just given for a single test cut into sections it would be expected that, except for the fluctuations of sampling, the correlations
of the s's with themselves or with g should vanish; in case, however, that the tests a, b, c, ... are really different involving arithmetic, space, perception, verbal or other specific abilities, it might not be expected that the correlations of the specific abilities would be zero, but resolution into general and specifics is made in such a way by hypothesis that a person above the average in arithmetic ability is not above the average but is random in verbal ability (the correlation of the tests being due entirely to the fact that each gives some expression to and measure of the general intelligence of the testee); moreover, the person who is above average in g is quite random with respect to each specific ability, and conversely, when judged relative to the group tested. This randomness needs pondering in connection with the discussion of the linear combination of tests conserving g. Whether the linear combinations which conserve g are of any psychological importance is probably tied up with the question as to whether the specifics are of any psychological importance.9

The variance \( r^2_{ig} = 1 - r^2_{ig} \) of g left undetermined by a team t made up of tests a, b, c, ... consists of two parts, one due to the failure of the tests to be perfectly reliable, the other due to there being outstanding independent specifics of real psychologic content not yet represented in the team. If by repetition or enlargement the tests a, b, c, ..., without any other change in their psychologic significance, are made more and more reliable until we (for logical simplicity) consider them perfectly reliable, we shall find \( r_{ag}, r_{bg}, \ldots \) increasing with the reliability about as \( 1/\sqrt{r_{aa}}, 1/\sqrt{r_{bb}}, \ldots \), but even for perfect reliability \( r_{ag}, r_{bg}, \ldots \) cannot become 1 so long as any specific factor of real content remains in the tests, i.e., so long as an arithmetic test is arithmetic and a verbal test verbal, etc. Thus even with perfectly reliable tests \( r^2_{ig} \) will not vanish since Z cannot become infinite for the reason that none of the angles \( \alpha, \beta, \ldots \) can become zero.10 The addition of new tests fitting into the hierarchical system will reduce \( r^2_{ig} \) but it cannot be reduced to zero unless one of the tests is actually a pure (and perfectly reliable) test of g itself, and in that case the team t reduces to that test alone ignoring all others.11

In such considerations the question of the uniqueness of g, g', g, ... simply does not arise; the discussion has assumed that there were definite general intelligences relative to the group and that we had to measure them by tests assumed to belong to a hierarchical system. The question of linear combinations of scores was not involved, nor that of overlap of specifics. I originally introduced the transformation theory as an indication of the possibility that g was not unique because it seemed to me in my ignorance of psychological scoring that different methods of scoring might be equivalent to such linear combinations if not to something more upsetting, and because the easiest way to eliminate overlap analytically when a new test contains a new specific with parts of old specifics is by
linear combination of the new scores with old scores, and so on. The transformation theory was introduced to show what might happen, not what did happen, and I should be entirely willing to abandon it whether it conserves \( g \) or changes \( g \). But something relative to uniqueness must be said if a test arises which does not fit into a hierarchical system with those already adopted. If we start with 4 tests, \( a, b, c, d \), which are not hierarchical we may modify \( d \) into \( d' \) in such a way that \( a, b, c, d' \) are a hierarchical set, or we may modify \( c \) into \( c' \) so that \( a, b, c', d \) are hierarchical, etc.; but the orthocenters of \( a, b, c, d' \) and \( a, b, c', d \) will not be the same, at least if we keep the modified vectors in the same 4-space as the original ones, which would be the most natural procedure. Then if hierarchical sets of a large number of tests are built up consistent with the two sets \( a, b, c, d' \) and \( a, b, c', d \) they must lead to two different \( g \)'s because the projections of the two \( g \)'s on the 4-space must be different. Thus the uniqueness of \( g \) would depend on every one adopting the same or equivalent hierarchical sets. This might be considered as forced upon all investigators if the specifics were due to specific independent genetic factors, but otherwise it would hardly be forced and at any rate there is as yet no consensus on the matter among psychologists because there are those who do not subscribe to the principle of the hierarchical system and the single general factor which may be determined therefrom except for an outstanding variance \( r_{ij} \).

In closing with the acknowledgment that the remarks I here offer have been elaborated after conferences with Professors Spearman, Freeman, Thurstone, Holzinger and Schultz at the University of Chicago, June 19 and 23, I wish to absolve them from any errors I may have made.

3 The notation and line of thought of my article in these Proceedings, 14, pp. 283–291 (1928), will be followed.
4 We have a rotation of the \( k \) vectors \( s_1, s_2, \ldots \) into \( s'_1, s'_2, \ldots \) with or without mirror reflection.
5 In case there are only three tests the nine constants \( l_i, m_i, n_i \), may be expressed simply in terms of three independent numbers \( p, q, r \) to which any values may be assigned, viz., if \( C = 1 + p^2 + q^2 + r^2 \),

\[
\begin{align*}
C_1 &= 1 + p^2 - q^2 - r^2, & C_m &= 2pq + 2r, & C_n &= 2pr - 2g, \\
C_2 &= 2pq - 2r, & C_m &= 1 + q^2 - p^2 - r^2, & C_n &= 2qr + 2p, \\
C_3 &= 2pr + 2q, & C_m &= 2qr - 2p, & C_n &= 1 + r^2 - p^2 - q^2
\end{align*}
\]

to specify the rotations combined with interchange of \( a \) and \( b \) to allow for the mirroring. See Gibbs-Wilson, Vector Analysis, p. 343. The rotation will be infinitesimal if \( p, q, r \) are infinitesimal; in that case the higher powers of \( p, q, r \) may be neglected and the equations become \( l_1 = m_1 = n_1 = 1, m_1 = -l_2 = 2r, l_3 = -n_2 = 2q, n_3 = m_3 = 2p \) from which it is at once evident that no infinitesimal rotation can have its constants all positive. For infinitesimal rotations in higher dimensions see C. L. E.
The argument is simple. Equations (3) show that if there are to be no negative coefficients the \(k(k - 1)\) terms \(id_i, m_i, m_{ij}, n, m_{n}, \text{etc.}, \) must all vanish. This will be so if only one coefficient in each line, a different one for each line, should fail to vanish, say \(m_1 = 1, i_2 = 1, n_3 = 1, \) etc.; but this would merely reproduce the set \(a, b, c, \ldots\) in some other order such as \(b, a, c, \ldots\) and would fail to be a change so far as the group of tests was concerned. If, however, two or more coefficients in one line, say, \(i_1\) and \(m_1\) with \(i_1^2 + m_1^2 = 1,\) fail to vanish then \(id_i = m_i = 0\) and \(i_1 = m_i = 0,\) for every \(j\) other than \(1;\) thus if the tests \(a\) and \(b\) enter into \(a'\) neither of them enters into any of the other \(k - 1\) tests \(b', c', \ldots\) and those tests contain only \(k - 2\) of the original tests which would make them linearly dependent among themselves as is impossible—one of the original tests would have been lost. To put this another way one may note that the square of the determinant of the \(k\) coefficients of an orthogonal transformation (2) must be 1 by virtue of (3) and that the absence of two tests from \(k - 1\) of the combinations would make the determinant vanish.

There might be different groups of tests \(a, b, c, \ldots\) and \(a', b', c', \ldots\) not linear combinations each of the other, and not necessarily even containing the same number of tests, which would give the same \(g.\) Indeed if all we desired was the equality of \(g\) so far as it was determined by the tests we should merely equate the teams

\[
t = \frac{a \cot \alpha \csc \alpha + b \cot \beta \csc \beta + \ldots}{\sqrt{Z(1 + Z)}}
= \frac{a' \cot \alpha' \csc \alpha' + b' \cot \beta' \csc \beta' + \ldots}{\sqrt{Z'(1 + Z')}}
= t'.
\]

In a particular case we could not expect to find an absolute identity between the \(t\) and \(t'\) for each individual.

\(As r_{tw} = \sqrt{r_{wa}}\) is greater than \(r_{wu}\) we see that the test determines \(g\) better than it determines the result of a repetition of the test—just as a single element \(x\) of a set is a better representative of the mean of the set than of any other (random) element of the set. The form of \(t\) is not simply \((a + b + c)/3\) but that multiplied by \(\sqrt{r_{wa}}.\) This is because the three individual parts \(a, b, c\) have been standardized to unit variances. If we write \(a_z = a_{iz} + d_z\) where \(a_{iz}\) is the true score of testee \(x\) on the test \(a\) and \(d_z\) is his fluctuation, \(a_z^2 = a_{iz}^2 + d_z^2\) and \(r_{za} = a_{iz};\) by hypotheses \(a_{iz} = b_{iz} = c_{iz}\) but the fluctuations \(d_z, e_z, f_z\) being fortuitous will not be identical. The sum \(a + b + c\) will be \(3a_i + d + e + f\) and

\[
\left(\frac{a + b + c}{3}\right)^2 = \frac{a_{iz}^2 + 1}{3} \sigma_{a}^2 = \sigma_{a}^2 + \sigma_{d}^2 = 1.
\]

The factor \(\sqrt{r_{xa}/r_{wa}}\) is to bring the variance of the sum up to 1; but in practice one would simply compound the scores and then standardize to unit variance.

When \(g\) is determined from the team \(t\) the specifics have become of no importance because they have been eliminated on the average, \(r_{at}\) being that mean value of \(g\) which is the average of all persons who have specified scores on \(a, b, c, \ldots\); the value of \(g_z\) for different members of this specified group being \(r_{az}d_z = \sqrt{1 - r_{ia}}.\)

For \(Z\) see footnote (7). We have \(r_{iz}^2 = (1 + Z)^{-1}.

It is without practical significance to discuss teams of tests whose number \(k\) is comparable to the number \(n\) of persons tested, but it may be interesting just to note that even if we have \(k = n - 2\) tests, hierarchical and none of them \(g\) itself, \(g\) will not be determined and that if we have \(k = n - 1\) tests which are hierarchical one of them must be \(g\) itself or the resolution into general and specifics becomes impossible for lack of an additional dimension.