We have had the benefit of discussions with Drs. J. C. Slater and W. B. Nottingham.

3 Bernhard A. Rose, Ibid., current number.
4 H. E. Farnsworth, Ibid., 35, 1131 (1930); 40, 684 (1932).
5 C. Davisson and L. H. Germer, Ibid., 30, 705 (1927).
6 During the initial stages of outgassing, a copper surface becomes negative.

NUMBER OF OPERATORS OF PRIME POWER ORDERS CONTAINED IN A GROUP

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Let $H$ represent a Sylow subgroup of order $p^m$ contained in a given group $G$ and suppose that $H$ has a cross-cut with each of its other conjugates which is of the same order for every pair of such conjugates and is invariant thereunder. We shall first prove that $H$ must then have a common cross-cut with each of its other conjugates and that this common cross-cut is invariant under $G$. If two of these cross-cuts were distinct then each of them would transform the other into itself since they are both invariant under $H$. All the operators of $G$ which transform one of these cross-cuts into itself generate a subgroup of $G$ which involves no Sylow subgroup of order $p^m$ contained in $G$ that has a different cross-cut with $H$ since the cross-cuts of $H$ with its other conjugates are of the same order.

It results from what precedes that all the Sylow subgroups of order $p^m$ contained in $G$ would appear in sets such that every pair of these sets would have only $H$ in common and the sets would be characterized by the common cross-cuts which would appear therein. A Sylow subgroup of order $p^m$ which is distinct from $H$ would transform the common cross-cut of a set to which it does not belong into a subgroup of the set to which it belongs and also into a subgroup of the set to which it does not belong. As this subgroup would not appear in $H$ but would transform into themselves two distinct cross-cuts with $H$ we have arrived at a contradiction by assuming that these cross-cuts are distinct and hence the following theorem has been established: If all of the cross-cuts of a given Sylow subgroup with its other conjugates are of the same order and invariant under these conjugates
then all of these cross-cuts must be identical and this common cross-cut is invariant under the entire group.

If \( p^n \) is the order of this common cross-cut and if the number of these Sylow subgroups is \( 1 + kp \) then the number of the operators of \( G \) whose orders are powers of \( p \) is \( kp^{m+1} + p^m - kp^{a+1} \). Since this number must be divisible by \( p^m \) it results that \( k \) is necessarily divisible by \( p^{m-a-1} \). As a useful corollary of the given theorem it may be noted that if a Sylow subgroup of order \( p^m \) has a subgroup of index \( p \) in common with each of its other conjugates then the group contains an invariant subgroup of order \( p^{m-1} \) and the number of its operators whose orders are powers of \( p \) is \( kp^{m+1} + (1 - k)p^m \). Another corollary is as follows: If the Sylow subgroups of a given order contained in a group are abelian and if all the cross-cuts of one of these Sylow subgroups with its other conjugates are of the same order then these cross-cuts are identical and this common cross-cut is an invariant subgroup of the entire group.

When \( k < p \) each of the Sylow subgroups of order \( p^m \) contained in \( G \) must have a subgroup of index \( p \) in common with every other such subgroup and hence the number of the operators of \( G \) whose orders are powers of \( p \) must then be \( kp^{m+1} + (1 - k)p^m \). It should be noted that this theorem is also true when \( k = 0 \) as well as when \( k = 1 \). It is also true when \( k = p + 1 \) since \( H \) could then not have a cross-cut of index \( p \) under \( H \) with exactly \( p \) of its other conjugates. For if this were the case these \( p + 1 \) Sylow subgroups would be the only ones which would involve this cross-cut. This is impossible since \( p^2 \) is not divisible by \( p + 1 \) and hence the remaining Sylow subgroups would not have a similar property relative to other such subgroups.

If \( k = p \) it is possible that \( G \) involves a set of \( p^2 \) conjugates under \( H \) each of which has a cross-cut with \( H \) which is of index \( p^2 \) under \( H \). These cross-cuts must then be identical for if they were not identical they would constitute a set of \( p \) conjugates under \( H \) each of which would appear in \( p \) other conjugates of \( H \). One such conjugate would then also have a subgroup of index \( p^2 \), which does not appear in \( H \), in common with \( p \) of its other conjugates. This is impossible since not more than one such subgroup could be selected from each of \( p - 1 \) sets of subgroups of order \( p^m \) which are conjugate under \( H \). Hence the following theorem has been established: If a group \( G \) contains less than \( (1 + p)^2 \) Sylow subgroups of order \( p^m \) then these subgroups contain an invariant subgroup of \( G \) as their common cross-cut and the number of operators of \( G \) whose orders are powers of \( p \) is \( kp^{m+1} + (1 - k)p^m \) except that when \( k = p \) this number may also be \( p^{m+2} \).

When \( G \) contains \( (1 + p)^2 \) Sylow subgroups of order \( p^m \) three new possibilities present themselves as follows: (1) These subgroups need no longer have a common cross-cut, (2) these cross-cuts need then not have a com-
mon order for every pair of these subgroups, (3) the cross-cuts are not necessarily invariant under the entire group. To construct a system of groups in which each of these three possibilities is realized we let \( n \) be such that \( 2^n - 1 = p \) and form the direct product of two simply isomorphic groups of order \((2^n - 1)p\) which have separately an invariant subgroup of order \( 2^n \) and \( 2^n \) subgroups of order \( p \). The group thus obtained contains \((1 + p)^2\) Sylow subgroups of order \( p^2 \) and one of them has a cross-cut of order \( p \) with each of \( 2p \) of the others while its cross-cut with each of the remaining \( p^2 \) of these subgroups is the identity. The former of these cross-cuts are not invariant under this direct product. Hence the following theorem: There are groups which involve separately \((1 + p)^2\) Sylow subgroups of order \( p^m \) such that the cross-cuts of different pairs of these subgroups have different orders and some of these cross-cuts are non-invariant under these groups separately.

Suppose that \( G \) has exactly \((1 + p)^2\) Sylow subgroups of order \( p^m \) and the cross-cuts of \( H \) with some of its conjugates has the largest possible value, viz., \( p^2 \). It then has cross-cuts of index \( p \) with \( 2p \) of its conjugates and these cross-cuts must be two distinct subgroups of \( H \) since \( p^2 \) is not divisible by \( 1 + 2p \). Each of the \( p \) conjugates of \( H \) which have the same cross-cut with \( H \) must have a cross-cut of index \( p \) with \( p \) of the other conjugates of \( H \) which have a cross-cut with \( H \) of index \( p^2 \) under \( H \). It therefore results that all the cross-cuts of index \( p^2 \) under \( H \) which \( H \) has with \( p^2 \) of its conjugates must be the cross-cut of the two subgroups of index \( p \) under \( H \) which are its cross-cuts with \( 2p \) of its conjugates. Hence all of the former cross-cuts are identical and the following theorem has been established: If a group contains exactly \((1 + p)^2\) Sylow subgroups of order \( p^m \) which do not have a common cross-cut of order \( p^{m-1} \) then it involves an invariant subgroup of order \( p^{m-2} \) and the number of its operators whose orders are powers of \( p \) is \( p^{m+2} \).

If \( p^a \) is the order of the largest cross-cut which \( H \) has with one of its conjugates which are different from \( H \) and if \( p^{a+\beta} \) is the order of the largest subgroup of \( H \) which transforms this cross-cut into itself so that it has \( p^{m-\alpha-\beta} \) conjugates under \( H \) then each of these conjugates is invariant under a subgroup of \( G \) which involves Sylow subgroups of order \( p^{a+\beta} \). These Sylow subgroups can have no operators in common unless they appear in \( H \). Hence the total number of these Sylow subgroups which involve a complete set of such conjugate cross-cuts under \( H \) contain \( p^{m+\beta} \) operators whose orders are powers of \( p \). This proves the following theorem: If \( p^a \) is the order of the largest cross-cut which a Sylow subgroup of order \( p^m \) has with one of its other conjugates then the number of operators whose orders are powers of \( p \) contained in the Sylow subgroups of order \( p^{a+\beta} \), the largest subgroup which transforms such a cross-cut into itself, is \( p^m + \beta \), where \( p^{a+\beta} \) is the order of the largest subgroup of \( H \) which transforms this cross-cut into itself.
If $H$ has a cross-cut of order $p^a$ with one of its other conjugates and if $p^{a+\lambda}$ is the largest subgroup of $H$ which transforms this cross-cut into itself then this cross-cut is invariant under another subgroup of $G$ whose Sylow subgroups whose orders are powers of $p$ are of order $p^{a+\lambda}$ and these Sylow subgroups have no operators in common except such as appear in $H$. They also have no other operators in common with the given Sylow subgroups of order $p^{a+\lambda}$. By continuing this process until all the cross-cuts of order $p^a$ which $H$ has with some of its conjugates have been considered it results that the largest subgroups of $G$ which involve these cross-cuts invariantly contain
\[ p^m(p^\beta + p^\beta_1 + \ldots + p^\beta_\lambda - \lambda) \]
operators whose orders are powers of $p$, where $\lambda + 1$ is the number of distinct complete sets of conjugate subgroups of order $p^a$, contained in $H$, which are cross-cuts of $H$ with its conjugates. It should be noted that this formula is independent of the value of $\alpha$. When $\alpha = m - 1$ then $\beta = \beta_1 = \ldots = \beta_\lambda = 1$. If $H$ has no other cross-cuts with its conjugates then $\lambda + 1 = k$ and this formula reduces to $kp^{m+1} + (1 - k)p^m$, which agrees with the formula noted above.

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**EVIDENCES OF MAN'S ANCESTRAL HISTORY IN THE LATER DEVELOPMENT OF THE CHILD**

**By C. B. Davenport**

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It has long been known that man, in his foetal development, passes through many stages reminiscent of the course of the evolution of the human species. Thus, the embryonic tail, gill-slits, tubular heart, amphibian-like kidney, for example, are ancestral. It is less well appreciated that at birth the child is still far from an adult, in proportions of parts, and has still to pass through a series of changes corresponding to stages at which adult primates have become adult.

At birth of the child the chest is a cylinder of approximately circular cross-section. This is the shape of the chest of the foetal stage of other primates and many other mammals. It is the generalized, undifferentiated form of the chest. In burrowing quadrupeds, like moles and shrews, the depth of the chest becomes eventually much less than the breadth; in flying quadrupeds the depth also becomes less than the breadth, but not so extreme as in the burrowers. In aboreal quadrupeds the chest retains