ON THE ASYMPTOTIC FORMULAE OF RIEMANN AND OF LAPLACE

BY Aurel Wintner

The Johns Hopkins University, Department of Mathematics

Communicated December 5, 1933

The Riemann Case.¹—Let \( f(x) \) denote a function having a continuous second derivative in the finite range \( 0 \leq x \leq a \), and let \( t \) be a positive parameter. Perron² has proved that

\[
\int_0^\alpha f(\lambda) \exp \left( \pm itx^2 \right) dx = (\pi/2^\alpha)^{-1/2} (1 \pm it)f(0)t^{-1/2} + O(t^{-1}) \quad (1)
\]

as \( t \to + \infty \) and applied this formula for the determination of the asymptotic behavior of functions³

\[
c(t) = \int_\alpha^\beta \phi(\lambda) \exp \left[ it\omega(\lambda) \right] d\lambda \quad (2)
\]

where \( \omega(\lambda) \) is real-valued and not a constant and \( \omega(\lambda), \phi(\lambda) \) are, for instance, regular-analytic in the interval \( \alpha \leq \lambda \leq \beta \). Since the Bessel functions \( J_\nu \) are integrals \( c(t) \) of the type (2), there follows in particular from (1) the classical relation \( J_\nu(t) \sim (2/\pi)^{1/2} \cos (t - \pi \nu/2) = \pi/4) t^{-1/2} \).
The reduction of the asymptotic behavior of (2) to the normalized formula (1) is, however, possible only if the derivative \(\omega'(\lambda)\) of the frequency nowhere possesses a multiple zero, as is the case for \(J_p(t)\). The vicinity of a point \(\lambda = \lambda_0\) at which \(\omega'(\lambda_0) \neq 0\) yields to (2) a contribution which is but of the order of the remainder \(O(t^{-1})\) in (1), inasmuch as

\[
\int_0^a f(x) \exp(\pm itx)dx = O(t^{-1})
\]

(3)
in virtue of the second mean-value theorem. In the first part of the present note Perron's proof for (1) will be slightly modified and simultaneously extended in such a way that the treatment of (2) is possible even if \(\omega'(\lambda)\) does possess a multiple zero. In order to obtain this extension one clearly has to generalize (1) for the case where \(\exp(\pm itx^n)\) is replaced by \(\exp(\pm itx^n)\), \(n \geq 2\). The result will be seen not to be the same as that for \(n = 2\), although just as simple, viz.,

\[
\int_0^a f(x) \exp(\pm itx^n)dx = \exp(\pm \pi i/2n) \Gamma(1 + n^{-1})t^{-1/n} + O(t^{-2/n})
\]

(4)
where \(t \rightarrow + \infty, n \geq 2\). For \(n = 2\) one obtains precisely (1) inasmuch as \(2\Gamma(3/2) = \Gamma(1/2) = \pi^{1/2}\). For \(n = 1\), of course (4) is false as illustrated by examples as \(f(x) = 1\) or \(f(x) = x\). Since \(t^{-2/n} = O(t^{-1})\) for every \(n \geq 2\), it is clear from (4) and (3) that the contribution of the vicinity of a point \(\lambda = \lambda_0\) to the integral (2) always is absorbed by the \(O\)-term in (4) if \(\omega'(\lambda_0) \neq 0\). In the proof of (4) it will not be supposed that \(f(x)\) is analytic or that \(n(\geq 2)\) is an integer.

Suppose that \(f(x)\) possesses a continuous first derivative in the whole interval \(0 \leq x \leq a\) and a continuous second derivative in an arbitrarily narrow interval \(0 \leq x \leq b\) where \(0 < b \leq a\). On placing

\[
xg(x) = f(x) - f(0) - xf'(0) \quad \text{if} \quad 0 < x \leq a
\]

(5)
and \(g(x) = 0\) if \(x = 0\), it follows by a repeated application of the mean-value theorem of the differential calculus that \(g(x)\) possesses a continuous first derivative not only in the range \(0 < x \leq a\) but in the closed interval \(0 \leq x \leq a\) as well.

Furthermore

\[
\int_0^a xg(x) \exp(itx^n)dx = O(t^{-2/n}).
\]

(6)
For on applying partial integration the integral (6) may be written in the form

\[
g(a) \int_0^a x \exp(itx^n)dx = \int_0^a g'(x) \int_0^x y \exp(ity^n)dy \, dx
\]
or
\[ t^{-2/n} [g(a)H(at^{1/n}) - \int_0^a g'(x)H(xt^{1/n})dx] \]  
(6a)

where \( H(t) = \int y \exp (iy^n)dy \). Consequently, on writing \( y \) instead of \( y^n \),
\[ H(t^n) = \int_0^t y^{-1+2/n} \exp (iy)dy/n. \]  
(6b)

Hence \( H \) is a bounded function of \( t \) for every fixed \( n \geq 2 \). For if \( n > 2 \) so that the exponent of \( y \) in (6b) is \( < 0 \), then on applying the second mean-value theorem there follows the existence of the improper integral \( H(\pm \infty) \). If, on the other hand, \( n = 2 \) then \( |H| \leq 1 \) is obvious from (6b). Since \( H \) remains bounded as \( t \to + \infty \) and since \( g'(x) \) is continuous and therefore bounded in the finite range \( 0 \leq x \leq a \), the expression (6a) clearly is \( = O(t^{-2/n}) \) as stated under (6).

Furthermore,
\[ \int_0^a x \exp (itx^n)dx = O(t^{-2/n}). \]  
(7)

For on placing \( tx^n = y \), the integral (7) takes the form \( t^{-2/n} H(a^n t^n) \) where \( H \) is, as we saw, a bounded function.

Finally,
\[ \int_a^\infty \exp (itx^n)dx = O(t^{-2/n}). \]  
(8)

In order to prove (8) where \( a > 0 \) and \( n \geq 2 \), we first notice that the integral
\[ L_n(t) = \int_t^{+\infty} y^{-1+1/n} \exp (iy)dy \]
where \( n > 1 \) exists and is \( = O(t^{-1+1/n}) \) in virtue of the second mean-value theorem applied to a finite interval. Now it follows from \( L_n(t) = O(t^{-1+1/n}) \) that the integral (8) which may be written in the form \( t^{-1/n} L_n(a^n t)/n \) is \( = t^{-1/n} O(t^{-1+1/n}) = O(t^{-1}) \) if \( n > 1 \), hence a fortiori \( = O(t^{-2/n}) \) if \( n \geq 2 \).

On substituting (5) and (7) into (6) and combining the result with (8) one obtains
\[ \int_0^a f(x) \exp (itx^n)dx = f(0) \int_0^{+\infty} \exp (itx^n)dx + O(t^{-2/n}) \]  
(9)
inasmuch as the complementary integral (8) is absorbed by the \( O \)-term in (9). Since \( \Gamma(1+z) = z\Gamma(z) \) implies
\[ \Gamma(1 + n^{-1}) = n^{-1} \int_0^{+\infty} x^{-1+1/n} \exp (-x)dx = \int_0^{+\infty} \exp (-r^n)dr \]
where \( r^n = x \), there follows from Cauchy's theorem with the help of a standard appraisal the identity

\[
\int_0^{+\infty} \exp (ix^n)dx = \Gamma(1 + n^{-1}) \exp (\pi i/2n).
\]

Hence the asymptotic formula (4) is a consequence of (9) inasmuch as both members of (4) go over into conjugated complex values if one replaces \( i \) by \(-i\). Besides (4) is an asymptotic formula only if \( f(0) \neq 0 \) and otherwise a mere appraisal.

**The Laplace Case.**—Let \( \phi(\lambda) \) and \( \psi(\lambda) \) denote two regular-analytic functions in the interval \( \alpha \leq \lambda \leq \beta \), and let \( \psi(\lambda) \) be real-valued and nowhere negative in this interval. The question regarding the asymptotic behavior of the function

\[
C(t) = \int_\alpha^\beta \phi(\lambda) [\psi(\lambda)]'d\lambda \quad (t > 0, \quad \psi' \geq 0)
\]

for large values of \( t > 0 \) is known to be reducible to the corresponding question regarding the normalized functions

\[
C_n(t) = \int_0^\infty f(x) \exp (-tx^n)dx \quad (n = 1, 2, \ldots).
\]

It has been shown by Perron\(^6\) that the Laplace formula, which is analogous to (1), is for all these "functions of large numbers" (2a), (2b) but the first term of an infinite asymptotic development as illustrated by the divergent Stirling series in the theory of the \( \Gamma \)-function. Another proof has been given by Haar.\(^6\) The apparatus applied by Perron and Haar belongs to the theory of complex functions. It has been pointed out by the present author\(^7\) that the asymptotic development in question may be obtained solely by methods of real analysis. This proof consisted of successive partial integrations and used an existence theorem of Borel concerning non-analytic functions with derivatives of arbitrarily high order. The application of this Borel theorem has been then eliminated by Ikehara and by Wiener\(^8\) in the course of Wiener's Tauberian researches. The following way, suggested by the above treatment of Riemann's case, is still shorter and has the obvious advantage of being independent of a general theory.

First, if \( n > 0, \ m > 0, \ a > 0 \) and

\[
G(x) = \int_0^x y^{m-1} \exp (-y^n)dy
\]

then

\[
\int_0^a | G(as) - G(xs) | \ dx = O(s^{-1}), \quad 0 < s \to + \infty.
\]
For the integral (11) may be written in virtue of \( G(xs) \leq G(as) \), \( 0 \leq sx \leq sa \), in the form
\[
\int_0^a dx \int_0^{as} y^{m-1} \exp \left( \frac{-y^n}{y} \right) dy = s^{-1} \int_0^a dx \int_0^{as} y^{m-1} \exp \left( \frac{-y^n}{y} \right) dy
\]
where the factor of \( s^{-1} \) remains bounded when \( s \to + \infty \), inasmuch as

\[
\int_0^+ \infty K(x) dx \text{ where } K(x) = \int_x^+ \infty y^{m-1} \exp \left( \frac{-y^n}{y} \right) dy
\]
clearly exists.

Let \( f(x) \) be a function possessing an \( m \)-th continuous derivative \( (m > 0) \) in the interval \( 0 < x \leq a \). On placing
\[
x^{m-1} g(x) = f(x) - \sum_{k=1}^{m} x^{k-1} f^{(k-1)}(0) / \Gamma(k) \text{ if } 0 < x \leq a \quad (12)
\]
and \( g(x) = 0 \) if \( x = 0 \), there follows from the mean-value theorem of the differential calculus the existence and the continuity of the first derivative of \( g(x) \) not only in the range \( 0 < x \leq a \) but in the closed interval \( 0 \leq x \leq a \) as well. Furthermore,
\[
\int_0^a x^{m-1} g(x) \exp \left( \frac{-tx^n}{x} \right) dx = O(t^{-m+1/n}) \quad (n > 0) \quad (13)
\]
as \( t \to + \infty \). For the integral (13) takes by partial integration the form
\[
g(a) \int_0^a y^{m-1} \exp \left( \frac{-ty^n}{y} \right) dy - \int_0^a g'(x) \int_0^x y^{m-1} \exp \left( \frac{-ty^n}{y} \right) dy \ dx,
\]
or, on using the notation (10) and the definition \( g(0) = 0 \),
\[
t^{-m/n} g(a) G(at^{1/n}) - t^{-m/n} \int_0^a g'(x) G(x t^{1/n}) dx
\]
\[
= t^{-m/n} \int_0^a g'(x) [G(at^{1/n}) - G(x t^{1/n})] dx.
\]
Hence (13) is a consequence of (11) inasmuch as \( |g'(x)| \) is continuous and therefore bounded in the finite interval \( 0 \leq x \leq a \).

Furthermore, if \( a > 0, k > 0, n > 0, m > 0 \) then
\[
\int_0^+ \infty x^{k-1} \exp \left( \frac{-tx^n}{x} \right) dx = O(t^{-(m+1)/n}). \quad (14)
\]
For let \( M = M(t) \) denote the maximum of \( x^{k-1} \exp \left( -x^{n/2} \right) \) in the infinite range \( T \leq x < + \infty \) where \( T = at^{1/n} \) so that
for arbitrarily large values of the exponent $p$. The integral (14) is
\[
M(t) = O(t^{-p})
\]
for arbitrarily large values of the exponent $p$. The integral (14) is
\[
t^{-k/n} \int_0^a x^{k-1} \exp(-tx^n)dx
\]
inasmuch as the complementary integrals (14) are absorbed by the $O$-term.

On substituting (12) into (13) and combining the result with (14) one obtains
\[
\int_0^a f(x) \exp(-tx^n)dx = \sum_{k=1}^m t^{-k/n} f^{(k-1)}(0) \Gamma(k/n) + O(t^{-(m+1)/n})
\]
inasmuch as the complementary integrals (14) are absorbed by the $O$-term. Now on placing $tx^n = y$, the factor of $f^{(k-1)}(0) \Gamma(k/n)$ in the last formula proves to be $t^{-k/n} \Gamma(k/n)/n$. Hence
\[
\int_0^a f(x) \exp(-tx^n)dx = \sum_{k=1}^m t^{-k/n} f^{(k-1)}(0) \Gamma(k/n)/n \Gamma(k) + O(t^{-(m+1)/n})
\]
if $f(x)$ possesses an $m$-th continuous derivative. Consequently
\[
\int_0^a f(x) \exp(-tx^n)dx \sim \sum_{k=1}^\infty t^{-k/n} f^{(k-1)}(0) \Gamma(k/n)/n \Gamma(k)
\]
if all derivatives of $f(x)$ exist, e.g., if $f(x)$ is regular-analytic. This is the desired asymptotic development. It may be noted that $n(>0)$ need not be an integer. Furthermore, the derivatives of order $k > 1$ need not exist without a small neighborhood $0 \leq x \leq b$ of $x = 0$ where $b < a$.

1 Cf. B. Riemann, Gesammelte Werke, 246–248 (1876) (and, on the other hand, 400–406). Cf. also G. H. Hardy, Quarti. J. Math., 44, 1–40 (1913), and 242–263 where further references also are given.


5 O. Perron, Münchener Sitzungsberichte, 191–220 (1917).

