twigs and an abundance of cone scales, is closer to the modern Brazilian than it is to the Chilean species.

The flora is in no sense a rain forest, however, but a mixture of mesophytic and drier soil types. Thus there are a good many Sapindaceae, a fair number of Leguminosae and various other things indicating that while the rainfall was greater and better distributed and the temperature more genial than at present, the country was not completely forested. A more detailed analysis of the probable conditions is reserved for the final account of this flora. Another fact, already foreshadowed in earlier studies, is the lack of resemblance to African floras. The Miocene flora of Patagonia, as far as it is known, is typically American. This tends to discredit those students who would derive the Patagonian mammals from African ancestors, and it also furnishes no comfort to those who are inclined to believe in any of the current hypotheses of drifting continents.

1 Berry, Edward W., Johns Hopkins University Studies in Geology, No. 6, 185-252, pls. 1-9 (1925).

ON CERTAIN INVARIANTS OF CLOSED EXTREMALS

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Morse has classified closed extremals by what we shall call indices of periodicity. See reference 3, Ch. III. We propose in this paper to present briefly some facts about the determination of the indices of periodicity of a closed extremal and some necessary conditions on these indices in terms of a generalized Poincaré rotation number.

1. Index of Periodicity.—We are studying a calculus of variations problem in the usual parametric form with an integrand which is positive, of class \( C^3 \), homogeneous and positively regular. The space of the dependent variables is a regular, orientable, analytic \((n + 1)\)-dimensional manifold. The non-orientable case is readily reducible to the orientable case.

We suppose there are closed extremals, of which \( g \) of length \( \omega \) is one.

We first map \( g \) and its neighborhood on the \( x \) axis and its neighborhood in the Euclidean space \((x, y_1, \ldots, y_n)\) by a transformation with period \( \omega \) in \( x \). When the integral of our problem is transformed in the corresponding manner, the \( x \) axis is an extremal. We denote the segment \( 0 \leq x \leq \omega \) of the \( x \) axis by \( \gamma \).
We write the transformed integral in non-parametric form so that \( x \) is the variable of integration. The Jacobi equations, in the space \((x, \eta)\), are then

\[
\frac{d}{dx} \Omega'_{\eta_i} - \Omega_{\eta_i} = 0 \quad (i = 1, \ldots, n).  \tag{1}
\]

See reference 3, Ch. I, ¶ 4. We are interested also in the modified Jacobi equations

\[
\frac{d}{dx} \Omega'_{\eta_i} - \Omega_{\eta_i} + \lambda \eta_i = 0 \quad (i = 1, \ldots, n).  \tag{2}
\]

See reference 3, Ch. II, ¶ 4.

The index of periodicity of \( g \) is the number of negative values of \( \lambda \) for which the modified Jacobi equations have a non-null periodic solution of period \( \omega \). We count a value of \( \lambda \) a number of times equal to the number of such solutions which are linearly dependent. See reference 3, Ch. III.

The order of degeneracy of \( g \) is the number of linearly independent solutions of the Jacobi equations which have the period \( \omega \). If the order of degeneracy is zero, we call the extremal \( g \) non-degenerate. In this paper we shall suppose the extremal \( g \) is non-degenerate.

A point \( x = b \) on the axis of \( x \) is a conjugate point of the distinct point \( x = a \) of index \( r \) if exactly \( r \) linearly independent solutions of the Jacobi equations vanish at \( x = a \) and also at \( x = b \). In counting conjugate points on an interval, we shall count each conjugate point a number of times equal to its index.

2. Order of Concavity.—Let \( G \) be the family of solutions

\[
\eta_i = \gamma_i(x) \quad (i = 1, \ldots, n)
\]

of the Jacobi equations joining those points in the \( n \)-planes \( x = 0 \) and \( x = \omega \) whose coordinates \((\eta)\) are the same. When \( g \) is non-degenerate, the solutions of \( G \) depend linearly on \( n \) linearly independent solutions of the Jacobi equations, which may be represented as the columns of the matrix

\[
||z_{ij}(x)|| \quad (i, j = 1, \ldots, n).
\]

Corresponding to a solution \( \eta_i = \eta_i(x) \), \((i = 1, \ldots, n)\), of the Jacobi equations, we set

\[
\Omega_{\eta'i}(\eta, \eta') = \chi_{\eta}^{\eta'}(x) \quad (i = 1, \ldots, n) \tag{3}
\]

The quadratic form* \( K(u) = a_{\mu\nu}u_{\mu}u_{\nu} \quad (\mu, \nu = 1, \ldots, n) \) \( \tag{4'} \)

in which

\[
a_{\mu\nu} = z_{\mu}(0)[\chi_{\eta'}^{\eta}(x)]^{\frac{\pi}{2\omega}}_{z=0} \quad (i, \mu, \nu = 1, \ldots n) \tag{4''}
\]
is termed the concavity form of \(g\). It is a symmetric form. Its index\(^5\) and nullity\(^4\) are independent of the matrix \(\|z_\gamma(x)\|\) chosen as above. We term its index the \textit{order of concavity} of \(g\). Its nullity is the index of \(x = \omega\) as a conjugate point of \(x = 0\). See reference 3, Ch. III, §10, for a discussion of the concavity form.

If \(x = \omega\) is not conjugate to \(x = 0\), we may interpret the concavity form geometrically in the following manner. Let \(S\) be the manifold of points, in the space \((x, \eta)\) of the solutions of the Jacobi equations, which lie on solutions belonging to \(G\) and passing through points \((u)\) in the \(n\)-planes \(x = 0\) and \(x = \omega\) for which

\[
u_1^2 + \ldots + u_n^2 = 1.\tag{5}\]

This manifold \(S\) is roughly a tube enclosing \(\gamma\). The exterior normals to \(S\) at point \((u)\) in the \(n\)-plane \(x = 0\) and point \((u)\) in the \(n\)-plane \(x = \omega\) lie in a 2-plane through the \(x\) axis. Let \(\theta\) be the angle between these normals, measuring \(\theta\) from the normal at \(x = 0\) and counting that sense of rotation positive which leads from the positive \(x\) axis to either normal. Then for points on (5) the concavity form \(K(u)\) and the angle \(\theta\) satisfy the relation

\[
r(u) \sin \theta = K(u),\tag{6}\]

where \(r(u)\) is a positive continuous function of the variables \((u)\).

The following theorem is fundamental. See reference 3, Ch. III, §11.

**THEOREM 1.** The index of periodicity of \(g\) is equal to the number of conjugate points of \(x = 0\) on \(\gamma\) plus the order of concavity of \(g\).

The extremal \(g\) traced \(m\) times is a closed extremal. We shall denote it by \(g^m\). We shall refer to the index of periodicity of \(g^m\) as the \(m\)th index of periodicity of \(g\) and denote it by \(T_m\). We shall call the order of concavity of \(g^m\) the \(m\)th order of concavity of \(g\).

We shall refer to the segment \(0 \leq x \leq m\omega\) of the \(x\) axis as \(\gamma^m\).

3. **Index of Repletion.**—If \(m > 1\) and \(x = \omega\) and \(x = (m - 1)\omega\) are not conjugate to \(x = 0\), the matrices

\[
\|u_{ij}(x)\|\quad \|w_{ij}(x)\|\quad (i, j = 1, \ldots, n)
\]

exist with columns which are unique solutions of the Jacobi equations satisfying the conditions\(^†\)

\[
u_{ij}(0) = 0 \quad w_{ij}(0) = \delta_i^j\quad (i, j = 1, \ldots, n).
\]

\[
u_{ij}(\omega) = \delta_i^j \quad w_{ij}([m - 1]\omega) = 0
\]

Then we shall call the quadratic form

\[
R(z) = [\zeta^{\nu}(\omega) - \zeta^{\nu}(0)]z_i z_j\quad (i, j = 1, \ldots, n)\tag{7}\]
the $m$th repletion form of $g$ and its index the $m$th index of repletion of $g$. Its nullity is the index of $x = m\omega$ as a conjugate point of $x = 0$.

The following theorem brings out the meaning of the indices of repletion.

**Theorem 2.** If $x = \omega, x = (m - 1)\omega$, and $x = m\omega$ are not conjugate to $x = 0$, the number of conjugate points of $x = 0$ on $\gamma^m$ exceeds the sum of the number on $\gamma$ and the number on $\gamma^{m-1}$ by the $m$th index of repletion of $g$.

4. **Determination of Indices of Periodicity.**—We let

$$
\| p_{ij}(x) \ q_{ij}(x) \| \quad (i, j = 1, \ldots, n),
$$

whose columns are solutions of the Jacobi equations, be the base of solutions satisfying the initial conditions

$$
\begin{align*}
p_{ij}(0) &= \delta_i^j, & q_{ij}(0) &= 0 \\
\xi_{ij}^p(0) &= 0, & \xi_{ij}^q(0) &= \delta_i^j
\end{align*}
$$

We define the matrix $A$ by the equation

$$
A = \begin{bmatrix} p_{ij}(\omega) & q_{ij}(\omega) \\ \xi_{ij}^p(\omega) & \xi_{ij}^q(\omega) \end{bmatrix} \quad (i, j = 1, \ldots, n). \tag{8}
$$

We shall use the matrix $A$ first in these two theorems.

**Theorem 3.** If for each positive integer $m$, $g^m$ is non-degenerate and $x = m\omega$ is not conjugate to $x = 0$, then the $m$th order of concavity and the $m$th index of repletion of $g$ are determined when $A$ is known.

**Theorem 4.** If for each positive integer $m$, $g^m$ is non-degenerate and $x = m\omega$ is not conjugate to $x = 0$, then the $m$th index of periodicity of $g$ is determined when $A$ and the number of conjugate points of $x = 0$ on $\gamma$ are known.

We let $W$ be a non-singular $n$-square matrix of constants. We set

$$
\begin{align*}
P &= \| p_{ij}(\omega) \| & Q &= \| q_{ij}(\omega) \| \\
Z^p &= \| \xi_{ij}^p(\omega) \| & Z^q &= \| \xi_{ij}^q(\omega) \|
\end{align*}
$$

We set

$$
\begin{align*}
R &= WPW^{-1} & S &= WQW^* \\
Y^p &= W^{-1*}Z^pW^{-1} & Y^q &= W^{-1*}Z^qW^* \tag{9'}
\end{align*}
$$

We show under the conditions of Theorem 4 that the problem of determining the $m$th index of periodicity of $g$ depends only on the number of conjugate points of $x = 0$ on $\gamma$ and on invariants depending on the sets $(R, S, Y^p, Y^q)$, where $W$ is arbitrary subject to the conditions in the preceding paragraph.

5. **The Frequency Number $\mu$.**—Any two solutions $\eta_i = v_i(x)$ and $\eta_i = z_i(x), (i = 1, \ldots, n)$, of the Jacobi equations satisfy a relation

$$
v_i(x)\xi_{ij}^p(x) - z_i(x)\xi_{ij}^q(x) = \text{constant} \quad (i = 1, \ldots, n)
$$
and if this constant is zero, the two solutions will be termed \textit{conjugate}. See reference 4, p. 626. There are at most \( n \) linearly independent solutions of the Jacobi equations in a set of mutually conjugate solutions. All the solutions which are linearly dependent on \( n \) mutually conjugate solutions will be said to form a \textit{conjugate family}, and any set of \( n \) linearly independent solutions belonging to a conjugate family will be said to form a \textit{conjugate base} of the conjugate family. A determinant whose columns are the solutions of a conjugate base will be termed a \textit{focal determinant} of the conjugate family. The zeros of a focal determinant are isolated and of integral order from 1 to \( n \). Any two focal determinants of a conjugate family vanish at the same points and to the same order. These points will be called \textit{focal points} of the conjugate family. The order of vanishing of a focal determinant at a point is equal to its nullity at the point and will be referred to as the \textit{order} of the focal point. In counting focal points of a conjugate family on an interval, each focal point will be counted a number of times equal to its order. This material is treated at length for a more general problem in reference 3, Ch. III.

Let \( F \) be a conjugate family with \( \nu \) focal points on the interval \( a < x \leq a + m\omega \), \((m = 1, 2, \ldots)\). We define the frequency number \( \mu(F,a) \) of \( g \) with respect to the conjugate family \( F \) and the initial point \( x = a \) as the limit

\[
\lim_{m \to \infty} \frac{\nu}{m}
\]

if this limit exists.

\textbf{Theorem 5.} \textit{The frequency number of} \( g \text{ with respect to a conjugate family and an initial point exists and is independent of the conjugate family and the initial point.}

The number \( \mu \) whose existence is affirmed in this theorem depends only on \( g \) and will be called the \textit{frequency number} of \( g \). The number \( \frac{\omega}{\mu} \) is a generalization of the so-called Poincaré rotation number of a point transformation on a closed curve. See reference 7. Hedlund has shown the existence of this frequency number in the case \( n = 1 \) and has shown its relation to the problem of determining the indices of periodicity \( T_m \) in that case. See reference 8.

The following theorem gives a necessary condition on the indices of periodicity of \( g \) in terms of its frequency number.

\textbf{Theorem 6.} \textit{The limit}

\[
\lim_{m \to \infty} \frac{T_m}{m}
\]

\textit{exists and is equal to the frequency number} \( \mu \) of \( g \).

6. \textit{Necessary Conditions on the Frequency Number.}—With the extremal
g we shall associate 2n numbers, $\rho_1, \ldots, \rho_{2n}$, which we shall call the multipliers of $g$. These numbers are the roots of the characteristic equation of the matrix $A$.

There are no multipliers equal to 0. The non-degeneracy of $g$ implies there are no multipliers equal to 1. We assume for convenience that there are no multipliers equal to $-1$.

The 2n numbers

$$\alpha_j = \delta_j + i\epsilon_j \quad (j = 1, \ldots, 2n) \quad (12)$$

defined by the relations

$$\log \rho_j = \delta_j + i\epsilon_j \quad (\pi < \epsilon_j \leq \pi) \quad (j = 1, \ldots, 2n) \quad (13)$$

we shall term the exponents of $g$. Compare with reference 10.

Through application of the theory of lambda-matrices we have established these theorems.

**Theorem 7.** If none of the exponents of $g$ is pure imaginary, the frequency number of $g$ is an integer.

**Theorem 8.** If the ratio of each pure imaginary exponent of $g$ to $2\pi i$ is a rational number, the frequency number of $g$ is rational.

* We adopt the convention that a repeated subscript indicates summation with respect to that subscript.

† The symbol $\delta_i$ is 1 when $i = j$ and 0 when $i \neq j$.

‡ If $B$ is a matrix, we denote its conjugate by $B^*$ and its inverse by $B^{-1}$.

1 Morse, "Closed Extremals," these PROCEEDINGS, 15, 866-859 (1929).


6 Reference 3, Ch. III, ¶ 2.


