cussed above would also involve modifications in the equations of macroscopic relativistic mechanics, since these would allow no creation or destruction of energy from the point of view of a local observer. These modifications might prove of interest for the problems of relativistic cosmology. The possible nature of the changes which could be introduced will be discussed in a following note.

2 See the collection of data given by Rutherford, Chadwick and Ellis, *Radiations from Radioactive Substances*, Cambridge (1930).
8 See the discussion of Poincaré, *Science and Hypothesis*, London (1905).

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**THE DIRAC EQUATION IN PROJECTIVE RELATIVITY**

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1. In this note we discuss the extension of the Dirac equation to general relativity. In order to have equations which are automatically invariant with respect to gauge as well as coordinate and spin transformations, we use the method of projective relativity. This theory is physically equivalent to the original general relativity theory of an Einstein gravitational and a Maxwell electromagnetic field.

We are led to a class of equations of the Dirac type. One of these reduces exactly to the Dirac equation of a charged particle in special relativity. This equation is identical with the one given by Schrödinger\(^1\) and therefore equivalent to the one given by Fock.\(^2\) The others contain extra terms which correspond to physical situations in which the field of a dipole is superposed on the field of the charge. Such extra terms, with the charge 0, have been proposed by Pauli in order to explain the properties of the neutron.\(^3\) The class thus contains equations first proposed by Schouten and Van Dantzig,\(^4\) in which an extra term appeared. One of these equations coincides with an equation proposed by Pauli,\(^5\) and may be considered as the simplest one in the projective notation, while the one without the extra term is simpler in the affine notation (equation (3.3)).
In the following discussions the weight $\frac{1}{4}$ is assigned to spinors, which are to be interpreted as Dirac wave functions in order that expressions of the type,

\[ J^\alpha = \gamma^\alpha_{\beta\gamma} A^\beta \psi^\gamma, \quad J^{\alpha\beta} = s^{\alpha\beta}_{\gamma\delta} \psi^\gamma \psi^\delta \]

which may represent the current vector and other invariants of the Dirac theory, shall behave as absolute projective vectors and tensors. We are using the notations of previous notes on spinors in these PROCEEDINGS.6

The most general equation satisfying the conditions of linearity and invariance is

\[ L^\alpha \psi_{\alpha} + N\psi = 0 \quad (1.1) \]

where the comma refers to projective differentiation, and $L^\alpha$ and $N$ are arbitrary spinors. The projective differentiation is with respect to the projective connection whose components $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the second kind formed from the fundamental projective tensor $\gamma_{\alpha\beta}$. Since any other projective connection associated with $\gamma_{\alpha\beta}$ would differ from this one by a projective tensor, we do not lose any generality by this restriction. The additional terms which would arise from using another projective connection are accounted for by the arbitrary matrix $N$.

Since the coefficients of the only terms involving $\frac{\partial \psi}{\partial x^\alpha}$ are $L^i$, these four matrices must reduce to the Dirac matrices in the special relativity case. This is accomplished by identifying them with matrices $\gamma^i$ which satisfy

\[ \frac{1}{4}(\gamma^i \gamma^j + \gamma^j \gamma^i) = \frac{1}{4} g^{ij}.1 \quad (1.2) \]

where $g^{ij} = \gamma^{ij}$ are the contravariant components of the fundamental metric tensor.7 In order that $\gamma_{\alpha\beta}$ shall be the fundamental projective tensor of P. R. we choose the matrices $\gamma^\alpha$ and $\gamma$ according to the footnote7 below rather than according to S. P. R. Equation (1.1) becomes

\[ \gamma^i \psi_{,i} + L^\alpha \psi_{,\alpha} + N\psi = 0. \quad (1.3) \]

Using the arbitrariness of $N$, this may be written as

\[ \gamma^\alpha \psi_{,\alpha} = \gamma_0 M\psi \quad (1.4) \]

where $M$ is a spinor which we will determine by requiring that equation (1.4) reduce exactly to the usual Dirac equation in the special relativity case.

2. In view of the formula for the projective derivative of a spinor (1.4) is equivalent to

\[ \gamma^\alpha \left( \frac{\partial \psi}{\partial x^\alpha} + K_{\alpha} \psi \right) = \gamma_0 M\psi \quad (2.1) \]
where (P. D. S., p. 88, equation (4.4))

\[ K_a = \Gamma^\alpha_{\beta \gamma} \gamma^\beta + \gamma^\beta \Gamma^\alpha_{\beta \gamma} + \frac{1}{2} \eta_a \] and \( \eta^A_B = \gamma^E_A \frac{\partial \gamma_{EB}}{\partial x^A}. \) (2.2)

From P. R., page 44, we have

\[
\begin{align*}
\Gamma^\alpha_{\nu \gamma} &= \gamma^\mu_{\nu \mu} \\
\Gamma^\alpha_{00} &= 0 \\
\Gamma^i_{jk} &= \{j_k\} + \varphi^i_j \varphi_k + \varphi^i_k \\
\Gamma^i_{jk} &= -\varphi^i_j \Gamma^i_{jk} + \frac{1}{2} \left( \frac{\partial \varphi^i_j}{\partial x^k} + \frac{\partial \varphi^i_k}{\partial x^j} \right)
\end{align*}
\] (2.3)

where \( \{j_k\} \) are the Christoffel symbols of the second kind formed from the \( g_{ij} \) and \( \varphi_{a \beta} = \frac{1}{2} \left( \frac{\partial \varphi_a}{\partial x^\beta} - \frac{\partial \varphi_\beta}{\partial x^a} \right). \)

Substituting (2.3) in (2.2) we obtain

\[ K_0 = \varphi^i s_{ij} \] (2.4)

where \( s_{a \beta} = \frac{1}{2} (\gamma^a \gamma^\beta - \gamma^\beta \gamma^a) \)

\[ \gamma^i K_i = -2 \gamma_0 K_0 + \gamma^i \varphi_i K_0 + \gamma^i S_i \] (2.5)

where

\[ S_i = (\gamma_k - \varphi_k \gamma_0) \left( \{i_k\} \gamma^i + \frac{\partial \gamma^k}{\partial x^i} \right) + \frac{1}{2} \eta_i + \gamma_0 \frac{\partial \gamma_0}{\partial x^i}. \] (2.6)

Equation (2.1) becomes

\[ \gamma^i \left( \frac{\partial \psi}{\partial x^i} + S_i \psi - I \varphi_i \psi \right) + \gamma_0 (I - K_0) \psi = \gamma_0 M \psi \] (2.7)

where \( I \) is the index of \( \psi \) and we have made use of the relation

\[ \gamma_0 = \gamma_0 \gamma^\alpha = \gamma^0 + \varphi_i \gamma^i. \] (2.8)

The spinor \( M \) may be expanded in terms of our basis of spinor matrices thus:

\[ M = a \cdot 1 + a_\alpha \gamma^\alpha + a_{a \beta} s_{a \beta} \] (2.9)

where \( a \) is a projective scalar, \( a_\alpha \) a projective vector and \( a_{a \beta} \) an antisymmetric projective tensor.

The quantities \( a, a_\alpha, a_{a \beta} \) must be built up out of the physical quantities which belong to the theory, taking account of the prescribed transformation properties.

3. In the special relativity case a coordinate system and spin-frame may be found such that \( \{j_k\} = 0, \eta_i = 0, \) and \( \gamma_0 \) and \( \gamma^i \) are constant.
matrices; hence in this coördinate and spin-frame \( S_i = 0 \). The Dirac equation in this coördinate system and spin-frame is

\[
\gamma^i \left( \frac{\partial \psi}{\partial x_i} - \frac{2\pi i e}{\hbar c} \sqrt{\frac{2}{\kappa}} \phi \psi \right) - \frac{\pi mc}{\hbar} \psi = 0. \tag{3.1}
\]

Hence it is clear that (2.7), in which \( I, a, a_a \) and \( a_{ab} \) are arbitrary, represents a class of equations of the Dirac type. If we choose

\[
a_{ab} = b\phi_{ab}, \quad a = I, \quad \text{and} \quad a_a = 4\mu\phi_a \tag{3.2}
\]
equation (2.7) becomes

\[
\gamma^i \left( \frac{\partial \psi}{\partial x^i} + S_i \psi - I\phi \psi \right) = \mu \psi + \gamma_0 (b + 1) \phi_{ij} \psi \tag{3.3}
\]
which reduces to the Dirac equation of special relativity if

\[
I = \frac{2\pi i e}{\hbar c} \sqrt{\frac{2}{\kappa}} \mu = \frac{\pi mc}{\hbar}, \quad \text{and} \quad b = -1. \tag{3.4}
\]

Equation (3.3) agrees with the general relativity form of the Dirac equation given by Schrödinger,\(^1\) in all spin-frames in which \( \gamma_{AB} \) and \( \gamma_0 \) are constants. Our \( S_i \) is the traceless part of Schrödinger's \( \Gamma_i \).

If \( b \neq -1 \) we obtain an equation with the additional term \( \gamma_0 \phi \psi_{ij} \) multiplied by \( b + 1 \), which may be interpreted physically as the interaction of the electron with the field of a dipole. Such a term has been suggested by Pauli\(^3\) as a possible explanation of the neutron. It also appears in the equation given by Pauli\(^5\) as the general relativity generalization of the Dirac equation. However, in this work it is multiplied by a factor containing \( \sqrt{\kappa} \), the Einstein gravitational constant which makes it negligible. To obtain Pauli's result we set \( b = 0 \).

4. The affine form of the Dirac equation of general relativity is (3.3) with the choice of constants given in (3.4). The same equation in projective notation is

\[
\gamma^\alpha \psi, \alpha = \gamma_0 (I + 4\mu\phi_\alpha \gamma^\alpha - \phi_{\alpha\beta} \phi^{\alpha\beta}) \psi. \tag{4.1}
\]

This may be written as

\[
\gamma^\alpha (\psi_\alpha - \phi_\alpha \psi_0) + 2\gamma_0 K_0 = \mu \psi. \tag{4.2}
\]

The equation corresponding to \( b = 0 \) is

\[
\gamma^\alpha (\psi_\alpha - \phi_\alpha \psi_0) = \mu \psi. \tag{4.3}
\]

This is a very natural one since it is a relation among the terms of the left side of the equation of spin displacement,

\[
\psi_\alpha - \phi_\alpha \psi_0 = 0 \tag{4.4}
\]
which corresponds to the projective displacement

$$X_{\sigma}^{\alpha} - \varphi_{\sigma}X_{\alpha}^{\sigma} = 0 \quad (4.5)$$

which is used to define the path of a charged particle in projective relativity (P. R., p. 46).

1 E. Schrödinger, "Dirac'sches Electron im Schwerefeld I," Berl Ber., 11, 105 (1932).
4 Schouten and Van Dantzic, Ann. Math., 34, 304 (1933).
6 We use capital letters, $A, B, C$ ($= 1, 2, 3, 4$) for spin indices, small Roman letters $i, j, k$ ($= 1, 2, 3, 4$) for affine tensor indices and small Greek letters, $\alpha, \beta, \gamma \ldots$ ($= 0, 1, 2, 3, 4$) for projective tensor indices. Formulas and results of the following previous papers in these PROCEEDINGS are used: O. Veblen, "Spinors in Projective Relativity," 19, 979–989 (1933) (referred to as S. P. R.); O. Veblen and A. H. Taub, "Projective Differentiation of Spinors," 20, 85–92 (1934) (referred to as P. D. S.); J. W. Givens, "Projective Differentiation of Spinors," 20, 232 (1934).

The fundamental projective tensor of the projective relativity is related to the Einstein gravitational tensor $g_{ij}$ and the electromagnetic projective vector $\varphi_{\alpha}$ by the equations

$$\gamma_{ij} = g_{ij} + \varphi_{i} \varphi_{j}; \quad \gamma_{\alpha \beta} = \varphi_{\alpha}; \quad \varphi_{0} = 1 \quad (1)$$

of which $g_{ij} = \gamma_{ij}$ is a consequence. The field equations of projective relativity in affine form are (P. R., p. 52)

$$\psi_{i}^{\alpha} = 0, \quad R^{i} = -\frac{1}{2} g^{i} R = -2S^{i} \quad (2)$$

where the semicolon refers to covariant differentiation with respect to $g_{ij}$, and

$$\varphi_{i} = \frac{1}{2} \left( \frac{\partial \varphi_{i}}{\partial x^{j}} - \frac{\partial \varphi_{j}}{\partial x^{i}} \right) \quad (3)$$

The first set of field equations are the Maxwell equations for free space. They and the remaining field equations agree with those of the general relativity if we set

$$\varphi = \sqrt{\frac{\kappa}{2}} \chi_{i} \quad (4)$$

where $\kappa$ is the Einstein gravitational constant and $\chi_{i}$, the electromagnetic four-potential.

The signature of $\gamma_{\alpha \beta}$ must be $(+++--)$ or $(----)$ in order to give the right signature for $g_{ij}$. As Prof. Robertson has pointed out in a lecture which elucidated this whole question, equations (2) are such that changing the sign of the $g_{ij}$'s would require changing the sign of $\kappa$, and the signature $(+++--)$ for $\gamma_{\alpha \beta}$ is the one for which $\kappa$ is positive.

In P. R. the projective tensor satisfies the invariant condition $\gamma_{00} = 1$. This implies that the non-holonomic transformation of period two

$$d\vec{s} = dx^{0} + \varphi_{i} dx^{i}, \quad d\vec{s} = -dx^{i} \quad (5)$$

will carry it into a form in which
The choice of signature \((+++\cdots -)\) described in the above paragraph then implies that for any particular point of space-time the coordinates may be chosen so that \(g_{ij} = 0\) for \(i \neq j\) and \(g_{11} = g_{22} = g_{33} = -g_{44} = 1\).

On the other hand in S. P. R. the matrices \(\gamma_{\alpha}\) were determined in a fixed spin-frame for a projective tensor \(\gamma_{\alpha \beta}\) which for the above determination of gauge and coordinates is such that \(\gamma_{\alpha \beta} = 0\) if \(\alpha \neq \beta\) and \(\gamma_{00} = \gamma_{11} = \gamma_{22} = \gamma_{33} = -\gamma_{44} = -1\). A set of matrices \(\gamma^\alpha\) which in the given coordinate gauge and spin-frame give the proper signature for \(\gamma_{\alpha \beta}\) (i.e., the signature used in P. R.) are:

\[
\begin{align*}
\gamma^0 & = \frac{1}{2} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \\
\gamma^1 & = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & -i & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & i
\end{pmatrix} \\
\gamma^2 & = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} \\
\gamma^3 & = \frac{1}{2} \begin{pmatrix}
-i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & i
\end{pmatrix}
\end{align*}
\]

The spin-frame is determined as that in which \(\gamma_{AB}\) and \(\gamma^A\) have the form

\[
\begin{pmatrix}
0 & i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & -i & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

respectively.

This choice of fundamental matrices changes formula (4.9) of S. P. R. into

\[
\gamma_{AB} = -\gamma^B_A.
\]

\(8\) In the special relativity case there exists a coordinate system \(x\) in which \(g_{ij} = 0\) if \(i \neq j\) and \(g_{11} = g_{22} = g_{33} = -g_{44} = 1\). To such a coordinate system we apply the non-holonomic transformation (5) and obtain a coordinate system \(N\) in which the \(\gamma_{\alpha \beta} = 0\) if \(\alpha \neq \beta\) and \(\gamma_{00} = \gamma_{11} = \gamma_{22} = \gamma_{33} = -\gamma_{44} = 1\). Then in the coordinate system \(N\) the matrices \(\gamma^\alpha\) can be made to have the form given above by a suitable spin transformation. We call these constants \(\gamma^\alpha\). On transforming back to the coordinate system \(x\) we obtain

\[
\gamma^A = -\gamma^A, \quad \gamma_0 = \gamma_0 \quad \text{and} \quad \gamma_k = \varphi_k \gamma_0 - \gamma_k.
\]

Hence with this choice of spin-frame in the special relativity \(\gamma^A\) and \(\gamma_0\) are constants. Also in the non-holonomic system \(\gamma_{AB}\) is a constant, and since it is invariant under coordinate transformation it is constant in \(x\); hence \(\eta_i = 0\) in \(x\).