COMPLETE DIFFERENTIAL SYSTEMS

BY JOSEPH MILLER THOMAS

DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY

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Consider a complete system $S$ of linear homogeneous partial differential equations of the first order

$$X_{\alpha} u = 0 \quad (\alpha = 1, 2, \ldots, r). \quad (1)$$

Pfeiffer$^1$ and later Hoborski$^2$ have recently shown by lengthy demonstrations how (1) can be replaced by an equivalent system such that every system $S_\lambda$ ($\lambda = 1, 2, \ldots, r$) which comprises the first $\lambda$ equations of $S$ is complete. This form of the complete system might be called nested. That reduction to nested form is possible is trivial because, as is well known, mere algebraic solution of (1) puts it in jacobian form and every one of its subsystems is not only nested but also jacobian.

Both the authors$^3$ mentioned, however, have also generalized Jacobi's method of integration, designed for systems in jacobian form, so that it can be applied directly to systems which are merely nested. Their method can be described concisely as follows. The Poisson parentheses for the system $S_\lambda$ in nested form satisfy

$$(X_{\alpha}, X_\beta) = a^{\gamma}_{\alpha \beta} X_{\gamma} \quad (\gamma = 1, 2, \ldots, \lambda). \quad (2)$$

If $y^1, \ldots, y^\lambda$ constitute a fundamental set of solutions for $S_{\lambda-1}$, the application of (2) to $y^j$ gives

$$X_{\alpha}(\log X_\lambda y^j) = a^{\lambda}_{\alpha \lambda} \quad (\alpha = 1, 2, \ldots, \lambda - 1; \lambda \text{ fixed}). \quad (3)$$

Subtraction applied to (3) for two values of $i$ shows that $X_\lambda y^j/X_\lambda y^i$ is a solution of $S_{\lambda-1}$ and hence a function of the $y^i$'s alone.

The result of applying $X_\lambda$ to $u(y^1, \ldots, y^\lambda)$ is

$$(X_\lambda y^j) \frac{\partial u}{\partial y^i} \quad (4)$$

Accordingly the equation $X_\lambda u = 0$ is equivalent by division to an equation in the $y^i$'s, whose general solution is the general solution of $S_\lambda$.

If the original system is given in nested form, the foregoing method of integration is immediately applicable. From this standpoint it is of value. If reduction to nested form must precede the integration, however, preference of the nested to the jacobian form has to be justified, because reduction to the latter is so simple. If the reduction to nested form is without special properties, such justification is impossible. We shall indicate a special reduction to nested form which seems to present
particular interest. Let \( X, Y \) be two symbols of a jacobian system. If \( a \) is a function of the independent variables, we have

\[
(aX, aY) = (aXa)Y - (aYa)X.
\]

Hence, if the equations of a jacobian system are all multiplied by the same non-zero function, the system remains in nested form. In fact, every one of its subsystems constitutes a complete system. Therefore, this nested form does not depend upon the order in which the equations are written.

Now let \( a \) be a non-vanishing determinant of order \( r \) formed from the coefficients of the \( X \)'s. If (1) is solved for the corresponding set of \( r \) derivatives and the resulting equations are multiplied by \( a \), we have a simple reduction to nested form in the coefficient ring, whereas reduction to jacobian form by solution is performed in the coefficient field. In this way, cumbersome denominators and the resulting singularities may be avoided.

It would be of interest to know whether a complete system can be put in jacobian form in the coefficient ring.

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3 Hoborski’s treatment in this particular is much the simpler.

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**ON GENERAL MANIFOLDS**

**By Eduard Čech**

**The Institute for Advanced Study, Princeton**

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Let \( G \) be an abelian group. Let \( 0 \leq p \leq n \). A topological space \( R \) will be called an absolute orientable \( n \)-manifold of rank \( p \) over \( G \), if it satisfies the following axioms:

I. \( R \) is a bicom pact space.

II. \( \dim R = n \).

III. There exists an absolute \((n, R)\)-cycle over \( G \), which is not \( \sim 0 \).

IV. If \( S \neq R \) is a closed subset of \( R \), then every absolute \((n, S)\)-cycle over \( G \) is \( \sim 0 \) over \( G \).

V. If \( U \) is a given neighborhood of a given point \( x \) of \( R \), there exists a neighborhood \( V \subset U \) of \( x \) having the following property: If \( C^n \) is an