REGULAR SUBGROUPS OF A TRANSITIVE SUBSTITUTION GROUP

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It was incidentally stated in these PROCEEDINGS, 21, 469–472 (1935), that every transitive substitution group of degree \( p^m \), \( p \) being a prime number, contains a regular group of order \( p^m \). In the review of this article in the Jahrbuch über die Fortschritte der Mathematik, 61, 94 (1935), it was correctly stated that exceptions to this theorem appear for every value of \( p \). In what follows we aim to establish some theorems relating to conditions under which the given theorem is true, as well as conditions under which it necessarily fails. In the article to which we referred it was noted that every transitive group of degree \( p^m \) contains a transitive Sylow subgroup whose order is a power of \( p \) and this result will be assumed in what follows. Hence it will be necessary to consider only transitive groups of degree \( p^m \), and when the order of such a non-regular group is a power of \( p \) it obviously contains an intransitive invariant subgroup of index \( p \) whose transitive constituents are of degree \( p^{m-1} \), since it involves \( p \) systems of imprimitivity.

We shall first prove that every transitive group of degree \( p^2 \) contains a regular group. In view of the theorem noted above we may confine our attention in the proof of this theorem to the case when the order of such a group \( G \) is \( p^m \). It is known that \( G \) contains an invariant substitution of order \( p \) as well as a subgroup of index \( p \) whose transitive constituents are generated by the cycles of this substitution. Hence this subgroup of index \( p \) contains only one subgroup of order \( p \) which is invariant under \( G \). This includes the \( p \)th power of every substitution in \( G \) and is contained in every regular subgroup of \( G \). The number of the additional substitutions in such a subgroup is \( p^2 - p \) and hence \( G \) contains exactly \( p^{m-2} \) regular subgroups. That is, every transitive substitution group of order \( p^m \) and of degree \( p^2 \) contains \( p^{m-2} \) regular subgroups. The value of \( m \) may vary from 2 to \( p + 1 \) and hence the largest possible number of these subgroups is \( p^{p-1} \). A necessary and sufficient condition that these subgroups are invariant under \( G \) is that \( m \) does not exceed 3.
It was noted above that the intransitive subgroup of index \( p \) under \( G \) composed of all its substitutions which do not interchange any of the cycles of its substitution of order \( p \) which is invariant under \( G \) contains only one invariant subgroup of order \( p \) under \( G \). Hence it gives rise to a commutator subgroup of index \( p^2 \) under \( G \) which again contains only one invariant substitution under \( G \). As this commutator subgroup gives rise to a commutator subgroup of index \( p^3 \) under \( G \), etc., it results that the given intransitive subgroup contains one and only one invariant subgroup under \( G \) for each of the orders \( p, p^2, \ldots, p^{m-1} \). Each of these includes all of those which precede it. The invariant subgroup of order \( p^2 \) in this set is especially interesting since it involves only substitutions of degree \( p^2 - p \) besides the \( p \) invariant substitutions under \( G \).

Having established the existence of an infinite system of transitive groups of a degree which is of the form \( p^m, m > 1 \), such that each of them contains at least one regular subgroup, we proceed to consider infinite systems of transitive groups whose degrees are of this form but which do not involve any regular group as a subgroup. We shall first consider such a system of degree \( p^3 \). To construct this system we establish a \( p^p \) isomorphism between \( p \) transitive groups of degree \( p^3 \) and of order \( p^3 \) in such a way that the co-sets composed of substitutions of degree \( p^3 - p \) correspond, where the symbol \( p^p \) is employed instead of the more common but longer symbol \( (p,p,p, \ldots, p) \) times). If we extend the intransitive group thus formed by means of the product of the substitution of order \( p \) which merely interchanges the corresponding letters of the transitive constituents in the intransitive group and a substitution of degree \( p^3 - p \) from one of these constituents there results a group in which all of the additional substitutions are non-regular and hence this \( G \) contains no regular group as a subgroup. This \( G \) exists for every value of the prime number \( p \). In the special case when \( p = 2 \) such a group is of degree 8 and of order 32. It is given as number 14 in volume 1 (1935) of my Collected Works, page 389.

The groups of this system can clearly be employed in a similar manner to construct an infinite system of groups of degree \( p^4 \) such that no group of the system contains a regular subgroup. As this process can be continued indefinitely there results the following theorem: There exists a transitive group of degree \( p^m, m > 2 \) for every prime number \( p \), such that it does not contain a regular group as a subgroup but every transitive group of degree \( p^2 \) contains such a subgroup.

Suppose that \( G \) is a transitive group of degree \( p^m \) and that the transitive constituents of its intransitive subgroup of index \( p \) are regular. The subgroup of \( G \) composed of all its substitutions which omit a given letter is of degree \( p^m - p^{m-1} \) since the number of letters omitted by this subgroup is a divisor of the degree and a multiple of \( p^{m-1} \). As the construction of \( G \) comes under the general theory of constructing all the regular groups of
order \( p^m \) it results that \( G \) involves a regular subgroup. That is, every transitive group of degree \( p^m \) which involves an intransitive subgroup of index \( p \) whose transitive constituents are regular involves a regular subgroup. A simple illustration of this theorem is the group of order 8 and of degree 4. When the transitive group is of order \( p^{m+1} \) the substitutions which are of degree less than \( p^m \) are in the central of the intransitive subgroup of index \( p \). When the order of a transitive group of degree \( p^m \) has a maximal value it contains a regular subgroup since it is completely determined by its order, and every group of degree \( p^m \) whose order is a power of \( p \) is found in a Sylow group of the symmetric group of degree \( p^m \).

GENERALIZED INTEGRALS AND DIFFERENTIAL EQUATIONS

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The following theorems have been proved by the author:

1. Generalization of the Integral.—Theorem: In the interval \( X_0 \leq x \leq X_1 \) consider a function \( f(x) \), approximable by continuous functions and satisfying the inequality \( |f(x) - f(X_0)| < F \), and \( n \) continuous functions \( g_1(x), g_2(x), \ldots, g_n(x) \) of the bounded total variations \( G_1, G_2, \ldots, G_n \). Suppose that two functions \( a \) and \( b \) of \( f, x, g_1, g_2, \ldots, g_n \) in the domain

\[
|f - f(X_0)| \leq F, \quad X_0 \leq x \leq X_1, \quad |g_1 - g_1(X_0)| \leq G_1, \ldots, |g_n - g_n(X_0)| \leq G_n
\]

admit of continuous partial derivatives with respect to all arguments. Then, for every sequence of continuously differentiable functions \( f_k(x), k = 1, 2, \ldots \) satisfying the inequality \( |f_k(x) - f(X_0)| < F \) and tending to \( f(x) \) as \( k \) tends to \( \infty \), the limit

\[
L = \lim_{k \to \infty} \int_{X_0}^{x} a (f_k(x), x, g_1(x), \ldots, g_n(x)) \, db(f_k(x), x, g_1(x), \ldots, g_n(x))
\]

exists.

We write

\[
L = \int_{X_0}^{x} a(f(x), x, g_1(x), \ldots, g_n(x)) \, db(f(x), x, g_1(x), \ldots, g_n(x)).
\]

Remark. This integral is not identical to the Stieltjes integral. For the function \( f(x) = 0, 0 \leq x < 1, f(1) = 1 \), we have for instance

\[
\int_{0}^{1} f(x) \, df(x) = \int_{0}^{1} f^2(1) - f^2(0) = \frac{1}{2}.
\]