NOTE ON THE CURVATURE OF ORTHOGONAL TRAJECTORIES
OF LEVEL CURVES OF GREEN'S FUNCTION

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Although the shape of the level curves of Green's function for a plane region has been widely studied,\(^1\) the shape of their orthogonal trajectories seems to have been somewhat neglected. That is to say, although the images of concentric circles (Kreisbilder) in a smooth conformal map have been studied, the images of all lines through their common center (Radienbilder) have been much less studied. It is the object of the present note to devote some attention to the curvature of the latter.

We take as point of departure the following theorem:

**THEOREM 1.** Let \( R_1 \) be a region of the extended \((x, y)\)-plane containing the point at infinity in its interior; the boundary \( B_1 \) of \( R_1 \) is limited. Let there exist Green's function \( G(x, y) \) for \( R_1 \) with pole at infinity. Let \( \{T_1\} \) denote the set of orthogonal trajectories to the level curves \( C_r: G(x, y) = \log r \).

If \( R_1 \) is simply connected, the tangent to an arbitrary curve \( T_1 \) at an arbitrary point of \( T_1 \) must cut \( B_1 \). The set of asymptotes to all curves \( T_1 \) is identical with the set of lines through a single point \( D_1 \), the conformal center of gravity of \( R_1 \). Every straight line through \( D_1 \) cuts \( B_1 \).

If \( R_1 \) is multiply connected, the tangent to an arbitrary curve \( T_1 \) at an arbitrary point of \( T_1 \) must either cut \( B_1 \) or separate two components of \( B_1 \). The set of asymptotes to all curves \( T_1 \) is identical with the set of lines through a single point \( D_1 \), the conformal center of gravity of \( R_1 \). Every such straight line through \( D_1 \) either cuts \( B_1 \) or separates two components of \( B_1 \).

The curves \( T_1 \) are the lines of force corresponding to a distribution of positive matter on \( B_1 \), provided \( B_1 \) is sufficiently smooth. Under these circumstances, it is therefore intuitively obvious that the tangent to an arbitrary curve \( T_1 \) at an arbitrary point must cut \( B_1 \) if \( R_1 \) is simply connected and must either cut \( B_1 \) or separate two components of \( B_1 \) if \( R_1 \) is multiply connected. A formal proof of the corresponding fact, whether \( B_1 \) is smooth or not, is not difficult.\(^2\)

Let us fasten our attention on a particular curve \( T_1 \), and study the tangent to \( T_1 \) at a point \( P \) of \( T_1 \), where \( P \) is allowed to become infinite. The tangent to \( T_1 \) at \( P \) always passes through the center of curvature at \( P \) of the curve \( C \), on which \( P \) lies. When \( P \) becomes infinite, this tangent approaches the asymptote to \( T_1 \), and the center of curvature of \( C \) at \( P \) approaches \( D_1 \).\(^3\) Thus the asymptote to \( T_1 \) passes through \( D_1 \). There exists a curve \( T_1 \) whose asymptote has any given direction, by interpretation of curves \( T_1 \) as the images of radii in the smooth conformal map of a
circle, so the set of asymptotes to all curves $T_1$ is identical with the totality of lines through $D_1$.

The point $D_1$ seems first to have been introduced by Lucas,\(^4\) namely in the case that $B_1$ is defined by an equation $|p(z)| = \text{const.}$, where $p(z)$ is a polynomial in $z$. Here the point $D_1$ is simply the center of gravity of the roots of $p(z)$, and Lucas shows that the asymptotes of the curves $\arg[p(z)] = \text{const.}$ pass through $D_1$. In the case that $R_1$ is simply connected, the point $D_1$ was studied by Frank and Löwner,\(^8\) but not in connection with the properties expressed in Theorem 1. Frank and Löwner showed that $D_1$ is the common center of gravity of all the curves $C$, with suitably distributed positive spreads, so the name "conformal center of gravity" is appropriate for $D_1$. This relation of $D_1$ to the center of gravity of curves $C$, was extended by Walsh (loc. cit., §14) to multiply connected regions $R_1$. The fact that the asymptote to every curve $T_1$ either cuts $B_1$ or separates two components of $B_1$ can be proved both from the corresponding property for tangents to the curves $T_1$ and from the interpretation of $D_1$ as the center of gravity of the loci $C_r$.

By either an inversion in a circle or by a reciprocal transformation of the complex variable, Theorem 1 yields the following theorem, in which the term "circle" is used to include also straight lines.

**Theorem 2.** Let $R$ be a region of the extended $(x,y)$-plane, with boundary $B$. Let there exist Green's function $G(x,y)$ for $R$ with pole in the finite point $O$. Let $\{T\}$ denote the set of orthogonal trajectories to the level curves $C_r$: $G(x,y) = \log r$.

*If $R$ is simply connected, the circle through $O$ tangent to an arbitrary curve $T$ at an arbitrary point of $T$ must cut $B$. As a limiting case, every circle osculating a curve $T$ at $O$ must cut $B$. The totality of such osculating circles for all curves $T$ is precisely the set of circles through $O$ and through another fixed point $D$ depending on $O$. There exists no circle that separates both $O$ and $D$ from $B$.

*If $R$ is multiply connected, the circle through $O$ tangent to an arbitrary curve $T$ at an arbitrary point of $T$ must either cut $B$ or separate two components of $B$. As a limiting case, every circle osculating a curve $T$ at $O$ must either cut $B$ or separate two components of $B$. The totality of such osculating circles for all curves $T$ is precisely the set of circles through $O$ and through another fixed point $D$ depending on $O$. There exists no circle that separates both $O$ and $D$ from $B$.

In Theorem 2 the point at infinity may be an interior, exterior, or boundary point of $R$.

Theorem 2 is concerned with the orthogonal trajectories $T$ for a specific Green's function with assigned pole $O$. If $R$ is simply connected, the set of all such curves $T$ for all choices of $O$ is precisely the set of images in $R$
(under the smooth conformal mapping of $R$ onto the interior of the unit circle $T$) of the circles orthogonal to $T$.

There are certain special geometric situations under Theorems 1 and 2 that deserve explicit mention.

If the region $R_1$ has a center of symmetry, the point $D_1$ coincides with that center of symmetry. If $O$ is a center of symmetry for $R$, the corresponding point $D$ lies at infinity.

If the boundary of the region $R_1$ is convex, then $D_1$ lies exterior to $R_1$. Suppose $R$ has the property that whenever two distinct points $A'$ and $A''$ of $B$ are given, then $R$ contains all points of the arc $A'O A''$ of the circle through $O$, $A'$ and $A''$; under these circumstances $D$ lies exterior to $R$.

If the region $R_1$ has an axis of symmetry, then $D_1$ lies on that axis. If the region $R$ has an axis of symmetry $S$ on which $O$ lies, then $D$ also lies on $S$, and (Theorem 2) is so situated that the circle through $O$ and $D$ orthogonal to $S$ must either cut $B$ or separate two components of $B$. If the region $R$ is symmetric (anallagmatic) in a circle $S$ on which $O$ lies, then $D$ also lies on $S$; the circle through $O$ and $D$ orthogonal to $S$ must either cut $B$ or separate two components of $B$.

Theorems 1 and 2 have application to the study of coefficients of the mapping function in a smooth conformal map. Let the function

$$w_1 = F(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \ldots$$  \hspace{1cm} (1)

map the region $|z| < 1$ smoothly (schlicht) onto the region $R_1$ of Theorem 1 in the $w_1$-plane. Then $D_1$ is the point $w_1 = b_0$. Let the function

$$w = \frac{1}{w_1} = f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$  \hspace{1cm} (2)

map $R_1$ and $|z| < 1$ smoothly onto the region $R$ of Theorem 2 in the $w$-plane; we do not assume that the point at infinity is not an interior point of $R$, so $f(z)$ must be meromorphic but not necessarily analytic throughout the region $|z| < 1$. The point at infinity is an interior point of $R$ when and only when $w_1 = 0$ lies interior to $R_1$.

From equations (1) and (2) we find $a_2 = -b_0$, and for the region $R$ of course $w = 1/b_0$ is the point $D$ corresponding to the point $O : w = 0$.

It is an obvious consequence of (1) that at least one point of $B_1$ satisfies the relation $|w_1 - b_0| \geq 1$, so at least one point of $B$ satisfies the relation $\left|\frac{1}{w} + a_2\right| \geq 1$.

It is a consequence of Theorem 1 that $B_1$ is cut by the line through $w_1 = b_0$ perpendicular to the line joining $w_1 = b_0$ to the origin $w_1 = 0$. Consequently, under the conditions of Theorem 2, the boundary $B$ is cut by the circle $\left|w + \frac{1}{2a_2}\right| = \frac{1}{2|a_2|}$. In particular we have
We add two further remarks, applications of Theorem 2 to the study of the smooth conformal map \( w = f(z) \) of the interior of \( T: |z| = 1 \) in the \( z \)-plane onto the region \( R \) of the \( w \)-plane.

Each coaxial family of circles \( K \) through two points \( z_1 \) and \( z_2 \) of the \( z \)-plane with \( |z_1| < 1 \), corresponds to a set of curves in \( R \) whose osculating circles at the point \( w = f(z_1) \) form a coaxial family; each of these osculating circles cuts the image in \( R \) of every circle which lies interior to \( T \), which separates \( z_1 \) from \( T \), and which is orthogonal to the family \( K \).

Let \( Q \) be either a circle which lies completely interior to \( R \), or an analytic arc or curve each of whose osculating circles lies interior to \( R \), or more generally an analytic arc or curve in \( R \) whose osculating circle at no point \( O \) (considered variable) passes through the corresponding point \( D \) of Theorem 2; then the image of \( Q \) in the \( z \)-plane is an arc or curve none of whose osculating circles is orthogonal to \( T \). This remark is analogous to Carathéodory's remark that if \( R \) is convex, every line segment interior to \( R \) has as its image in the \( z \)-plane an arc each of whose osculating circles cuts \( T \).³

² Walsh, loc. cit., §8.
³ Walsh, loc. cit., Theorem 11.
⁴ Jour. de l'École polytechnique, 28, 1–33 (1879).
⁶ Whether or not \( R \) is convex, every line segment in the \( w \)-plane through the point \( w = f(0) \) has as its image in the \( z \)-plane a curve whose osculating circle at the point \( z = 0 \) cuts the circle \( |z| = 1/2 \.)