THE IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

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In the present note I state certain theorems on the irreducible representations of the symmetric group in the hope that they may lighten the labors of those carrying out theoretical investigations in nuclear physics. A detailed discussion of these results will, it is planned, appear later in the American Journal of Mathematics.

THEOREM 1. Denoting by \( \{ \lambda_1, \lambda_2, \ldots, \lambda_x \} \) the aggregate of characters of that irreducible representation \( D(\lambda_1, \lambda_2, \ldots, \lambda_x) \) of the symmetric group on \( n \) letters which is associated with the partition \( \lambda_1 + \lambda_2 + \ldots + \lambda_x = n \) of \( n \), \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_x > 0 \) and by \( \alpha = (\alpha_1, \ldots, \alpha_n) \) the class containing \( \alpha_1 \) unary cycles, \( \alpha_2 \) binary cycles, \( \alpha_3 \) ternary cycles and so on \((\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + n\alpha_n = n)\) then the character \( \{ \lambda_1, \ldots, \lambda_x \}_\alpha \) of \( D(\lambda_1, \ldots, \lambda_x) \) corresponding to any class \( \alpha \) which contains at least one cycle on \( p \) letters \((1 \leq p < n)\) may be obtained from the characters of the symmetric group on \( n-p \) letters by means of the formula

\[
\{ \lambda_1, \lambda_2, \ldots, \lambda_x \}_{\alpha} = \{ \lambda_1 - p, \lambda_2, \ldots, \lambda_x \}_{\alpha'} + \{ \lambda_1, \lambda_2 - p, \lambda_3, \ldots, \lambda_x \}_{\alpha'} + \ldots + \{ \lambda_1, \ldots, \lambda_x - p \}_{\alpha'}
\]

where \( \alpha' \) is the class of the symmetric group on \( n-p \) letters containing one less cycle on \( p \) letters than the corresponding class \( \alpha \) of the symmetric group on \( n \) letters. In using this formula it is understood that when the subtraction of \( p \) forces a reversal in the normal descending order this normal order is restored by interchanging the two elements which are in reversed order, increasing the one which was on the left by unity and decreasing the one which was on the right by unity, and prefixing a minus sign to the \( \{ \ldots \} \). E.g., \( \{3, 5, 2\} = -\{4, 4, 2\} \). It is an immediate consequence that \( \{ \ldots a, b, \ldots \} = 0 \) if \( b = a + 1 \) since \( \{ \ldots a, b, \ldots \} = -\{ \ldots b - 1, a + 1, \ldots \} = -\{ \ldots a, b, \ldots \} = 0 \). Similarly \( \{ \ldots a, b, c, d, \ldots \} = 0 \) if \( c = a + 2 \) or if \( d = a + 3 \) and so on. Furthermore zeros occurring at the end are dropped and any \( \{ \ldots \} \) containing a negative number at the end is dropped. The particular case of this theorem when \( p = 1 \) is due to Schur and has been the most useful in the construction of the character tables of the symmetric group. In the complete form of statement all characters of the symmetric group on \( n \) letters may be calculated from the, supposed known, characters of the symmetric groups on \( m < n \) letters save the characters of the class con-
taining one cycle on \( n \) letters. Fortunately for the force of the theorem these characters are trivially evident; all being zero save \( \{ n - k, 1, 1, \ldots 1 \} \) which equals \((-1)^k\), \( k = 0, 1, \ldots n - 1 \). Furthermore a repeated application of the theorem reduces the determination of the characters of a class for which \( \alpha_1 \neq 0 \) to that of the dimensions (= characters of the unit class) of representations of the symmetric group on \( \alpha_1 \) letters; and for these the convenient formula of Frobenius is available. E.g., suppose we wish to calculate the characters of the symmetric group on \( n = 20 \) letters corresponding to the class containing \( \alpha_1 = 12 \) unary cycles and \( \alpha_3 = 1 \) cycle on 8 letters. As an illustration consider the representation \( D(9, 6, 3, 2) \). Applying our theorem with \( p = 8 \) we obtain

\[
\{9, 6, 3, 2\}_{a'} = \{1, 6, 3, 2\}_{a'} + \{9, -2, 3, 2\}_{a'} + \\
\{9, 6, -5, 2\}_{a'} + \{9, 6, 3, -6\}_{a'}
\]

where \( a' \) is the unit class (i.e., that one consisting of the unit permutation, which contains 12 unary cycles) of the symmetric group on 12 letters. The two terms at the end are dropped (the second from the end because \( \{9, 6, -5, 2\} = -\{9, 6, 1, -4\} = 0 \)). The first term is dropped because the third member 3 of the partition is bigger by two units than the first member 1. The only remaining term, the second, transforms thus: \( \{9, -2, 3, 2\} = \{9, 2, -1, 2\} = \{9, 2, 1, 0\} = \{9, 2, 1\} \) and so the desired character is the dimension of the representation \( D(9, 2, 1) \) of the symmetric group on 12 letters, namely, \( 12! 10. 8. 2/11! 3! = 320 \).

**Theorem 2.** We denote by \( \Delta(\lambda_1, \lambda_2, \ldots \lambda_r) \) the reducible representation of the symmetric group on \( n \) letters which is associated with the partition \( (\lambda_1, \lambda_2, \ldots \lambda_r) \) of \( n \), i.e., the representation by means of the permutation matrices furnished by the cosets of that subgroup which permutes the first \( \lambda_1 \) letters, the second \( \lambda_2 \) letters, \ldots and so on amongst themselves. Then it is well known that \( \Delta(\lambda_1, \lambda_2) = D(\lambda_1, \lambda_2) + D(\lambda_1 + 1, \lambda_2 - 1) + D(\lambda_1 + 2, \lambda_2 - 2) + \ldots \), there being \( \lambda_2 \) terms in the summation on the right. The present theorem extends this result from the case \( k = 2 \) to a general value of \( k \). We shall merely state the theorem for \( k = 3 \), the general mode of procedure being then clear (we shall illustrate by means of an example for which \( k = 4 \)). Denote by \( \Sigma(\lambda_1, \lambda_2) \) the summation on the right of the expression for \( \Delta(\lambda_1, \lambda_2) \) and by \( (\mu, \Sigma(\lambda_1, \lambda_2)) \) the summation \( D(\mu, \lambda_1, \lambda_2) + D(\mu, \lambda_1 + 1, \lambda_2 - 1) + \ldots \) obtained by prefixing a \( \mu \) to each partition in \( \Sigma(\lambda_1, \lambda_2) \). Then

\[
\Delta(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, \Sigma(\lambda_2, \lambda_3)) + (\lambda_1 + 1, \Sigma(\lambda_2 - 1, \lambda_3)) + (\lambda_1 + 1, \\
\Sigma(\lambda_2, \lambda_3 - 1)) + (\lambda_1 + 2, \Sigma(\lambda_2 - 2, \lambda_3)) + (\lambda_1 + 2, \Sigma(\lambda_2 - 1, \lambda_3 - 1)) \\
+ (\lambda_1 + 2, \Sigma(\lambda_2, \lambda_3 - 2)) + \ldots
\]
Thus, for $n = 5$,

$$
\Delta(2, 2, 1) = D(2, 2, 1) + D(2, 3) + D(3, 1, 1) + D(3, 2) + D(3, 2)
+ D(4, 1) + D(4, 1) + D(5)
= D(2, 2, 1) + D(3, 1, 1) + 2D(3, 2) + 2D(4, 1) + D(5)
$$

the same provisions as to reversal of order still holding.

Similarly, for $n = 4$,

$$
\Delta(2, 1, 1) = D(2, 1, 1) + D(2, 2) + 2D(3, 1) + D(4)
$$

and, for $n = 3$,

$$
\Delta(1, 1, 1) = D(1, 1, 1) + 2D(2, 1) + D(3).
$$

As an example of the method when $k = 4$ consider the representation
$\Delta(3, 2, 2, 1)$, of dimension 1680, of the symmetric group on 8 letters. For
the first group of terms we simply prefix a 3 to the partitions in the formula,
given above, for $\Delta(2, 2, 1)$. For the next group of terms, where we in-
crease the first member 3 of the partition to 4, we have to deal with (2, 1, 1)
twice and (2, 2) once (it being allowable to write all the derived partitions
of $n - (\lambda_1 + 1) = 4$ in normal descending order). For the next group,
in which we increase the 3 to 5 we have to deal with (2, 1) four times and
(1, 1, 1) once. For the next group, in which 3 is increased to 6 we have
to deal with (1, 1) three times and (2) twice; and then when the 3 is
increased to 7 we have to deal with (1) three times. Finally, when the 3 is
increased to 8 we have to deal with (0) once (bringing into light the fact
that the identity representation is contained once in each and every redu-
cible representation $\Delta(\lambda_1, \lambda_2, \ldots, \lambda_r)$). We obtain finally

$$
\Delta(3, 2^4, 1) = D(3, 2^4, 1) + D(3^2, 2^2) + 2D(3^2, 2) + 2D(4, 2, 1^2) +
3D(4, 2^2) + 5D(4, 3, 1) + 2D(4^2) + D(5, 1^3) + 6D(5, 2, 1) +
5D(5, 3) + 3D(6, 1^2) + 5D(6, 2) + 3D(7, 1) + D(8).
$$

An important consequence of the theorem is that there are no $D(\lambda_1, \lambda_2,
\ldots, \lambda_j)$ with $j > \kappa$ in $\Delta(\lambda_1, \lambda_2, \ldots, \lambda_r)$.

**Theorem 3.** We shall denote by $D(\lambda_1, \lambda_2, \ldots, \lambda_r) \times D(\mu_1,
\mu_2, \ldots, \mu_j)$ the direct product of the indicated irreducible representations
of the symmetric groups on $n = \lambda_1 + \lambda_2 + \ldots + \lambda_r$ and $m = \mu_1 +
\mu_2 + \ldots + \mu_j$ letters, respectively. This direct product is a representation
(reducible) of the symmetric group on $n + m$ letters. The number of times various
of these direct products contain the different irreducible representations
of the symmetric group on $n + m$ letters is found as follows. Let us
denote by $E_p(\lambda_1, \lambda_2, \ldots, \lambda_r)$ the sum

$$
E_p(\lambda_1, \lambda_2, \ldots, \lambda_r) = D(\lambda_1 + p, \lambda_2, \ldots, \lambda_r) +
\ldots + D(\lambda_1, \lambda_2, \ldots, \lambda_r + p)
$$
by $E_p, q(\lambda_1, \lambda_2, \ldots \lambda_\ell)$ the sum obtained by adding $p$ to one member of the partition and $q$ to another in all possible ways; thus

$$E_p, q(\lambda_1, \lambda_2, \lambda_3) = D(\lambda_1 + p, \lambda_2, \lambda_3) + D(\lambda_1 + q, \lambda_2, \lambda_3) + D(\lambda_1, \lambda_2 + p, \lambda_3 + q)$$

$$+ D(\lambda_1, \lambda_2 + p, \lambda_3 + q)$$

with the previous provisions as to reversals of normal descending order as before, and so on.

Then (a) $D(1) \times D(\lambda_1, \ldots \lambda_\ell) = E_4(\lambda_1, \ldots \lambda_\ell, 0)$. E.g., $D(1) \times D(2, 1) = E_4(2, 1, 0) = D(2, 1, 1) + D(2, 2) + D(3, 1)$.

(b) $D(2) \times D(\lambda_1, \ldots \lambda_\ell) = E_2(\lambda_1, \ldots \lambda_\ell, 0) + E_{11}(\lambda_1, \ldots \lambda_\ell, 0)$. E.g., $D(2) \times D(3, 1) = E_2(3, 1, 0) + E_{11}(3, 1, 0) = D(5, 1) + D(3, 3) + D(4, 2)$

$$+ D(4, 1, 1) + D(3, 2, 1).$$

(c) $D(3) \times D(\lambda_1, \ldots \lambda_\ell) = E_4(\lambda_1, \ldots \lambda_\ell, 0) + E_{21}(\lambda_1, \ldots \lambda_\ell, 0) + E_{111}(\lambda_1, \ldots \lambda_\ell, 0)$ and so on.

(d) $D(1, 1) \times D(\lambda_1, \ldots \lambda_\ell) = E_{11}(\lambda_1, \ldots \lambda_\ell, 0) + D(\lambda_1, \ldots \lambda_\ell, 1, 1) = E_{11}(\lambda_1, \ldots \lambda_\ell, 0, 0)$. E.g., $D(1, 1) \times D(3, 1) = E_{11}(3, 1, 0, 0) = D(3, 1^3) + D(3, 2, 1) + D(4, 1^2) + D(4, 2)$.

(e) $D(1, 1, 1) \times D(\lambda_1, \ldots \lambda_\ell) = E_{111}(\lambda_1, \ldots \lambda_\ell, 0, 0, 0)$.

(f) $D(2, 1) \times D(\lambda_1, \ldots \lambda_\ell) = D(\lambda_1, \ldots \lambda_\ell, 2, 1) + E_{21}(\lambda_1, \ldots \lambda_\ell, 0) + E_{111}(\lambda_1, \ldots \lambda_\ell, 0, 0)$.

It is hoped that these illustrations will point the method to the interested reader. The complete avoidance of the troublesome use of the character tables is very time saving.

* During 1936–37, guest member of the Institute for Advanced Study.

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**GROUPS OF ORDER LESS THAN $2^m$ HAVING $m - 1$ OR $m - 2$ INDEPENDENT GENERATORS**

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A group whose order is less than $2^m$ can obviously not have as many as $m$ independent generators. If it has $m - 1$ independent generators its order is at least as large as $2^{m-1}$, and when it is of this order it is abelian and of type $1^{m-1}$. If its order exceeds $2^{m-1}$ but is less than $2^m$ and it has $m - 1$ independent generators the number of prime factors in its order is $m - 1$ and hence its order is $3.2^{m-2}$, $m > 1$. When $m = 2$ it is the group