ON THE THEORY OF BEAMS RESTING ON A YIELDING FOUNDATION

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Introduction.—The purpose of this note is to advance the theory of the following problem: An elastic beam of infinite length, loaded inside a finite region, rests on a yielding foundation. It is asked how much the results of this theory are influenced by alterations of the assumptions concerning the physical nature of the foundation. As it is known, in engineering the assumption is made that at every point the deflection is proportional to the foundation pressure at this point and independent of the pressure at points distant from the considered point. This assumption, which is justified for a beam resting on water, has the advantage of leading to the solution of a linear differential equation. In reality the deflection at one point depends almost always on the pressure distribution along the whole beam and the question leads, as it will be seen, to an integro-differential equation.

On this problem a very interesting paper has recently been published by M. A. Biot. There the foundation is represented by a semi-infinite elastic body. Here it shall be shown that the formal results of that paper can be obtained in a different, more general way; more general in that it is shown to be possible to solve the problem explicitly for every kind of foundation whether elastic or having properties more difficult to describe, provided the deflection of the surface of the foundation due to a concentrated loading is known.

It will also be shown that under this assumption the problem of buckling of a beam of infinite length, resting on a yielding foundation and on an infinity of equidistant supports, can be solved completely. So far as known a treatment of this problem has been given only for the beam resting on water.

Statement of the Problem.—The differential equation for the deflection \( w \) of an elastic beam of constant stiffness \( E_b J \) is given by

\[
E_b J \frac{d^4 w}{dx^4} = p(x).
\]

There \( p(x) \) denotes the acting load. Supposing that the beam rests on a yielding foundation, \( p \) consists of the given load \( p_0 \) and the foundation pressure \( p_u \)
\[ p = p_0 - p_u. \] (2)

The deflection of the surface of the foundation at a point \( x \) due to a loading \( p_u(\xi) \, d\xi \) inside a region \((\xi, \xi + d\xi)\) may be denoted by \( dv(x) \). Then

\[ dv(x) = K \left( |x - \xi| \right) p_u(\xi) \, d\xi \] (3)

where \( K \) is a function depending on the nature of the foundation which may be determined experimentally or otherwise.

The deflection \( v(x) \) due to the whole foundation pressure is then

\[ v(x) = \int_{-\infty}^{\infty} K(|x - \xi|) p_u(\xi) \, d\xi. \] (4)

For reasons of continuity \( v(x) \) is equal to the beam deflection \( w(x) \). Furthermore \( p_u \) connected with \( w \) and the loading \( p_0 \) by equations (1) and (2). Thus from (4) the following integro-differential equation for \( w \) is obtained.

\[ w(x) = \int_{-\infty}^{\infty} K(|x - \xi|) \left[ p_0(\xi) - E_0 \frac{d^4w}{d\xi^4} \right] \, d\xi \] (5)

or differentiating both sides with respect to \( x \)

\[ \frac{dw}{dx} = \int_{-\infty}^{\infty} \frac{\partial K}{\partial x} \left( |x - \xi| \right) \left[ p_0(\xi) - E_0 \frac{d^4w}{d\xi^4} \right] \, d\xi. \] (6)

Equations (5) and (6) are the fundamental equations of the problem of the elastically supported beam of infinite length. It is possible to give an explicit solution of (6) without knowing \( K \), provided \( K \) has certain properties which are always fulfilled for physical reasons and which do not need to be stated here.

It may be remarked that working with equation (6) instead of (5) is suggested by the fact that for the case of a foundation consisting of a two-dimensional semi-infinite elastic body \( K \) becomes infinite for large values of its argument and hence also \( w \), whereas its derivatives stay finite.

The Solution.—Assuming for the sake of simplicity an even load distribution \( p_0(x) = p_0(|x|) \) we write

\[ p_0(x) = \int_{0}^{\infty} P(\lambda) \cos \lambda x \, d\lambda \] (7)

where \( P(\lambda) \) is given by

\[ P(\lambda) = \frac{1}{\pi} \int_{0}^{\infty} p_0(x) \cos \lambda x \, dx. \] (8)
Then we try a solution in the form

\[ w(x) = \int_0^\infty W(\lambda) \cos \lambda x d\lambda \]

\[ \frac{dw}{dx} = -\int_0^\infty \lambda W(\lambda) \sin \lambda x d\lambda \]

\[ \frac{d^4w}{dx^4} = \int_0^\infty \lambda^4 W(\lambda) \cos \lambda x d\lambda \]  

(9)

Introducing (7) and (9) into (6) gives

\[ -\int_0^\infty \lambda W(\lambda) \sin \lambda x d\lambda = \int_0^\infty \frac{\partial K(|x - \xi|)}{\partial x} \left\{ \int_0^\infty P(\lambda) \cos \lambda \xi d\lambda - E_0 J \int_0^\infty \lambda^4 W(\lambda) \cos \lambda \xi d\lambda \right\} d\xi \]  

(10)

and changing the order of integration in the right side of (10), a process which is easily justified,

\[ \int_0^\infty \lambda W(\lambda) \sin \lambda x d\lambda = \int_0^\infty [E_0 J \lambda^4 W(\lambda) - P(\lambda)] \]

\[ \int_{-\infty}^\infty \frac{\partial K(|x - \xi|)}{\partial x} \cos \lambda \xi d\xi d\lambda. \]  

(11)

Putting \( \xi - x = u \)

\[ \int_{-\infty}^\infty \frac{\partial K(|x - \xi|)}{\partial x} \cos \lambda \xi d\xi = \cos \lambda x \int_{-\infty}^\infty \frac{\partial K(|u|)}{\partial u} \cos \lambda u d\xi 

- \sin \lambda x \int_{-\infty}^\infty \frac{\partial K(|u|)}{\partial u} \sin \lambda u d\xi. \]

(13)

Because \( \frac{\partial K(|u|)}{\partial u} \) is an odd function the first term on the right in (13) vanishes and we have

\[ \int_{-\infty}^\infty \frac{\partial K(|x - \xi|)}{\partial x} \cos \lambda \xi d\xi = -\sin \lambda x \int_{-\infty}^\infty \frac{\partial K(|u|)}{\partial u} \sin \lambda u d\xi. \]

(14)

Placing

\[ \int_{-\infty}^\infty \frac{\partial K(|u|)}{\partial u} \sin \lambda u d\xi = k(\lambda) \]

(15)
(11) can be written
\[
\int_0^\infty \lambda W(\lambda) \left[ 1 + E_0 J \lambda^3 k(\lambda) \right] \sin \lambda x d\lambda = \int_0^\infty P(\lambda) k(\lambda) \sin \lambda x d\lambda
\]
(16)

This implies
\[
\lambda W(\lambda) = \frac{P(\lambda) k(\lambda)}{1 + E_0 J \lambda^3 k(\lambda)}
\]
(17)

and hence
\[
\frac{dw}{dx} = - \int_0^\infty \frac{P(\lambda) k(\lambda)}{1 + E_0 J \lambda^3 k(\lambda)} \sin \lambda x d\lambda.
\]
(18)

Thus provided \( K \) and consequently the Fourier transform \( k \) of its derivative is known, the solution of (6) is explicitly given by (18).

As an application we may take the case of a two-dimensional semi-infinite elastic foundation of the same breadth 2b as the beam, where the result is already known. Here
\[
K(|x - \xi|) = \frac{1}{\pi bE} \ln |x - \xi| + \text{const.}
\]
(19)

and consequently
\[
\frac{\partial K(|u|)}{\partial u} = \frac{1}{\pi bEu}.
\]
(20)

Hence from (15)
\[
k(\lambda) = \frac{1}{\pi bE} \int_{-\infty}^\infty \frac{\sin \lambda u}{u} du = \frac{1}{bE}
\]
(21)

and from (18)
\[
\frac{dw}{dx} = - \frac{1}{bE} \int_0^\infty \frac{P(\lambda)}{1 + \frac{E_0 J}{bE} \lambda^3} \sin \lambda x d\lambda.
\]
(22)

The Stability Problem.—When the beam is compressed in its plane, the loading \( p_0 \) is given by the stress component vertical to the plane which arises from a deflection. If the stress resultant is equal to \( P_0 k g \) this component is
\[
p_0 = - P_0 \frac{d^2 w}{dx^2}
\]
(23)
and (6) becomes
\[ \frac{d^2w}{dx} = - \int_{-\infty}^{\infty} \frac{\partial K(|x-\xi|)}{\partial x} \left[ P_0 \frac{d^2w}{d\xi^2} + E_bJ \frac{d^4w}{d\xi^4} \right] d\xi. \quad (24) \]

A solution of this equation can be obtained which satisfies the following conditions

\[ w(x_n) = 0, \left( \frac{d^2w}{dx^2} \right)_{x=x_n} = 0; \quad x_n = \pm nl, \quad n = 0, 1, 2, \ldots. \quad (25) \]

These conditions state that the beam is freely supported at the points \( x_n \).

Trying

\[ w = \sin \frac{m\pi x}{l}, \quad m = 1, 2, 3, \ldots. \quad (26) \]

(25) is satisfied and (24) becomes

\[
\frac{m\pi}{l} \cos \frac{m\pi x}{l} = \int_{-\infty}^{\infty} \frac{\partial K(|x-\xi|)}{\partial x} \left[ P_0 \left( \frac{m\pi}{l} \right)^2 - E_bJ \left( \frac{m\pi}{l} \right)^4 \right] \sin \frac{m\pi \xi}{l} d\xi \quad (27)
\]

\[
\cos \frac{m\pi x}{l} = \frac{m\pi}{l} \left[ P_0 - E_bJ \left( \frac{m\pi}{l} \right)^2 \right] \cos \frac{m\pi x}{l} \int_{-\infty}^{\infty} \frac{\partial K(|u|)}{\partial u} \sin \frac{m\pi u}{l} du \quad (28)
\]

and with (15)

\[
1 = \frac{m\pi}{l} \left[ P_0 - E_bJ \left( \frac{m\pi}{l} \right)^2 \right] k \left( \frac{m\pi}{l} \right). \quad (29)
\]

Hence the critical stress resultant is given by

\[
P_0 = \frac{1}{\frac{m\pi}{l} k \left( \frac{m\pi}{l} \right)} + E_bJ \left( \frac{m\pi}{l} \right)^2. \quad (30)
\]

With the help of this result the question which number \( m \) of half waves yields the smallest critical \( P_0 \) can be answered. It is obtained from

\[
\frac{\partial P_0}{\partial m} = \frac{l}{\pi dm} \left( \frac{1}{mk \left( \frac{m\pi}{l} \right)} \right) + 2m \left( \frac{\pi}{l} \right)^2 E_bJ = 0. \quad (31)
\]
For the example of the two-dimensional elastic foundation\textsuperscript{2} (21), (30) is

\[ P_0 = E_b J \left( \frac{m \pi}{l} \right)^2 + \frac{b El}{\pi m} \]  

(32)

and from (31)

\[ m_{\text{min}} = \frac{1}{\pi} \sqrt[3]{\frac{E_b l^3}{2E_b J}} \]  

(33)

whereas in the case of a beam resting on water it is\textsuperscript{2}

\[ P_0 = \left( \frac{m \pi}{l} \right)^2 E_b J + \frac{\eta l^2}{m^2 \pi^2}; \; m_{\text{min}} = \frac{1}{\pi} \sqrt[4]{\frac{\eta l^4}{E_b J}} \]  

(34)

Concluding Remarks.—From the foregoing lines it can be seen that the problem of a beam resting on a yielding foundation admits a simple solution if the beam is of infinite length.

The same problem is by far more complicated for a beam of finite length. Here some results have been obtained by Wieghardt.\textsuperscript{4} The present author would like to add that he has recently made some progress in this direction which he hopes to publish soon.

3 Timoshenko, St., Theory of Elasticity, McGraw-Hill, 88 (1934).

GROUPS HAVING A MAXIMUM NUMBER OF INDEPENDENT GENERATORS

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It is obvious that the number of operators in a set of independent generators of a given group $G$ cannot exceed the number of the prime factors of the order of $G$. When $G$ has a set of independent generators involving as many operators as there are prime factors in its order we shall say that it has a maximum number set of independent generators. All the operators of such a set are of prime order, for if one of them would be of composite order the cyclic group generated by it could be successively extended by the other operators in the given set of independent generators. The group