theory is readily applied to the calculus of variations with the usual positive, and positive regular integrand in parametric form.

1 Morse, "Functional Topology and Abstract Variational Theory," *Annals of Mathematics,* 38, 386–449 (1937). We refer to this paper by the letter M. Complete references are given in this paper and in the fascicule to appear later.

2 The following book will appear shortly: Seifert und Threlfall, *Variationsrechnung im Grossen. Theorie von Marston Morse.* Teubner, Berlin. This book is highly recommended. The authors begin with two axioms similar to our accessibility hypothesis, but referring to singular cycles. These axioms are satisfied when the critical values cluster at most at infinity and when the critical points are isolated. In this way the most important cases are treated in the simplest way. To obtain greater generality Vietoris cycles seem to be useful. In fact the present author has shown in 3 (following) that the accessibility hypothesis is not in general satisfied when ordinary cycles are used, even when \( f \) is of class \( C^\infty \) on regular analytic manifolds and when the critical values are finite in number.


---

**ON CRITERIA CONCERNING SINGULAR INTEGERS IN CYCLOTOMIC FIELDS**

**By H. S. Vandiver**

**DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF TEXAS**

Communicated June 24, 1938

As elsewhere\(^1\) a singular integer is defined as an integer \( \alpha \) in the field \( k(\zeta) \); \( \zeta = e^{2\pi i/l} \), \( l \) an odd prime, such that \( \alpha = a^t \) where \( a \) is an ideal in \( k(\zeta) \) which is not principal. Necessary conditions that an integer in \( k(\zeta) \) be singular were given by Takagi,\(^2\) and when the field \( k(\zeta) \) is properly irregular, that is to say, when the second factor of its class number is prime to \( l \); a necessary and sufficient condition was given by the writer.\(^3\) Here we shall give some other necessary conditions for singular integers in any irregular cyclotomic field. Based on a result of Kummer's the writer\(^4\) obtained the relation

\[
\prod_{r=1}^{k-1} \frac{[u/k]}{b(\zeta^r)} \sim 1, \quad (1)
\]

that is, the ideal on the left is principal, where \( k \) is an integer \( 1 < k < l \); \( rr_1 \equiv 1 \pmod{l} \); \( [s] \) is greatest integer in \( s \), and \( b \) is any integer in \( k(\zeta) \) and \( b(\zeta^r) \) is obtained from \( b(\zeta) \) by the substitution \( (\zeta/\zeta^r) \); and it follows if we assume that (Vandiver\(^3\) \( \alpha \) is singular and semi-primary, then

\[
\prod_{r=1}^{k-1} \frac{[u/k]}{\alpha(\zeta^r)} = \omega^j
\]
where \( \omega \) is an integer in \( k(\xi) \). As it stands this is a necessary condition that \( \alpha \) be singular, but we shall show that this can be transformed into criteria which do not involve \( k \). The method is an extension, and modification, of one employed previously by the writer in connection with units in certain cyclic fields. We shall find it convenient to employ power characters in \( k(\xi) \). Set

\[
\left( \frac{\theta}{p} \right) = \xi^{f(\theta)}
\]

and

\[
\frac{N(p) - 1}{l} \equiv \left( \frac{\theta}{p} \right) \pmod{p}
\]

where \( p \) is any ideal in \( k(\xi) \) prime to \( (\theta) \) and \( (l) \) with \( \theta \) an integer in \( k(\xi) \). Also put

\[
D_* = \sum_{d=1}^{l-1} d^s I(\alpha(\xi^d))
\]

and consider the sum

\[
\sum_{s=0}^{l-2} \sum_{d=1}^{l-1} d^s I(\alpha(\xi^d))
\]

where \( d \) is one of the integers in the set \( 1, 2, \ldots, l-1 \). This may be put in the form

\[
(l-1) I(\alpha^{d_1}) + \sum_{d \neq d_1} d_1^{l-1-s} d_1^{l-1-s} I(\alpha(\xi^d))
\]

whence

\[
-I(\alpha(\xi^d)) \equiv D_0 + d_1^{l-2} D_1 + d_1^{l-3} D_2 + \ldots + d D_{l-2},
\]

modulo \( l \). Applying this to (1) after setting \( \xi^m \) for \( \xi \) we have, if \( \mu = (l-1)/2, \)

\[
\mu(k - 1) D_0 m^{l-1} + \sum_r (mr_1)^{l-2} D_1 + \sum_r (mr_1)^{l-3} D_2 + \ldots + \sum_r (mr_1) D_{l-2} \equiv 0 \pmod{l}.
\]

Now set \( m = 1, 2, \ldots, l-1 \) in turn we obtain \( (l-1) \) congruences and we obtain by elimination

\[
D_0 \equiv \sum r_1^{l-s} D_{s-1} \equiv 0 \pmod{l}, \quad s = 2, 3, \ldots, l-1.
\]

Using the known relation

\[
\frac{(1 - k^i) b_i}{k^{i-1} b_i} \equiv \sum r_1^{l-i} \pmod{l},
\]

\[
b_1 = -1/2, \quad b_{2a+1} = 0, \quad a > 0; \quad b_{2a} = (-1)^{a-1} B_a.
\]
the B's being the numbers of Bernoulli, \( B_1 = 1/6, B_2 = 1/30, \) etc., and letting \( k \) be a primitive root of \( l \) then (2) gives

\[
D_0 \equiv b_{s+1} D_s \equiv 0 \pmod{l},
\]

(3)

\( s = 1, 2, \ldots, l - 2 \). Suppose that \( b_{2a} \not\equiv 0 \pmod{l} \) then (3) gives

\[
D_{2a-1} \equiv 0 \pmod{l},
\]

\[
\left( \prod_{d=1}^{l-1} \frac{\alpha(\zeta^d)}{\beta} \right)^{d^{2a-1}} = 1,
\]

for any \( \beta \) prime to \( l \) and \( N(\alpha) \). Hence by a known result we have the

THOREM: If \( \alpha(\zeta) \) is a singular integer in the field \( k(\zeta) \) and is also semiprimary; \( \zeta = e^{2i\pi/l}, l \) an odd prime and \( B_a \not\equiv 0 \pmod{l}, a < (l - 1)/2, \) then

\[
A_{2a-1} = \prod_{d=1}^{l-1} \left( \alpha(\zeta^d) \right)^{d^{2a-1}} = \sigma^l
\]

(4)

where \( \sigma \) is an integer in \( k(\zeta) \).

If in (4) we obtain an identity in \( \varepsilon^l \) by adding a suitable multiple of \((l^a - 1)/(1 - 1)\) and differentiate \((l - 2a)\) times and set \( v = 0 \) we obtain

\[
\left[ \frac{d^{l-2a} \log \alpha(\zeta)}{d\varepsilon^{l-2a}} \right]_{v=0} \equiv 0 \pmod{l}
\]

which gives a known criterion that, \( \alpha(\zeta) \) is singular.

We also note that

\[
A_{2a-1} = \prod_{c=0}^{l-2} \left( \alpha(\zeta^c) \right)^{c(2a-1)}
\]

where now \( r \) is a primitive root of \( l \) and also

\[
A_{2a-1}(\zeta^r)^{2a-1} = A_{2a-1}(\zeta^r)^l
\]

where \( \delta \) and \( \gamma \) are numbers in \( k(\zeta) \).

Additional relations of this type hold if we restrict ourselves to properly irregular cyclotomic fields. Here for any singular integer in \( k(\zeta) \), say, \( \omega(\zeta) \), we have

\[
\omega(\zeta)\omega(\zeta^{-1}) = \varepsilon r^l
\]

where \( \varepsilon \) is a unit and \( r \) an integer in \( k(\zeta) \). Treating this relation in a similar manner to that employed in connection with (1) we obtain

\[
\prod_{d=1}^{l-1} \left( \omega(\zeta^d) \right)^{d^{2a}} = \beta^l\delta
\]
for each $a$ in the set $1, 2, \ldots, (l - 3)/2$, such that $\delta$ is a unit in $k(\gamma)$ with $\delta(\gamma)^{ra} = \delta(\gamma)^l$, and $B_a \equiv 0 \pmod{l}$.

1 Vandiver, these Proceedings, 15, 203 (1929).

GROUPS OF DEGREE $n$ INVOLVING ONLY SUBSTITUTIONS OF LOWER DEGREES

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

Communicated July 7, 1938

Every transitive group of degree $n$ involves at least $n - 1$ substitutions of this degree but an intransitive group of degree $n$ does not necessarily involve any substitution whose degree is as large as $n$. Such intransitive groups exist only when $n > 4$ and there is obviously only one such group of degree 5, viz., the group formed by a 3, 1 isomorphism between the symmetric group of degree 3 and the group of degree 2. Whenever $n$ is odd and exceeds 3 it is clearly possible to construct similarly a group of degree $n$ which does not involve any substitution whose degree is as large as $n$ by dimidiating the dihedral group of degree $n - 2$ and the group of degree 2. Exactly half of the substitutions of this group are of degree $n - 1$ while the remaining substitutions thereof appear in the cyclic subgroup of degree $n - 2$ and are either of degree zero or of degree $n - 2$.

From the preceding paragraph it results that when $n$ is odd and exceeds 3 it is always possible to construct at least one group of degree $n$ which does not involve any substitution of this degree and has one transitive constituent of degree $n - 2$. We proceed to prove that such a group cannot be constructed when $n$ is an even number. If this were possible the transitive constituent of degree $n - 2$ of such a group would involve a subgroup of index 2 containing all its substitutions of degree $n - 2$ since none of its other substitutions would be of as large a degree as $n - 2$. Hence all of these other substitutions would be of degree $n - 3$ since this is the average number of letters in the substitutions of this constituent. As $n - 3$ is supposed to be odd each of these substitutions of degree $n - 3$ involves an odd cycle and an odd power of such a substitution would involve less than $n - 3$ letters but would appear among the given substitutions of degree $n - 3$. As this is impossible it results that a necessary and sufficient condition for each $a$ in the set $1, 2, \ldots, (l - 3)/2$, such that $\delta$ is a unit in $k(\gamma)$ with $\delta(\gamma)^{ra} = \delta(\gamma)^l$, and $B_a \equiv 0 \pmod{l}$.

1 Vandiver, these Proceedings, 15, 203 (1929).