THE GENERALITY OF FINITE ABSTRACT COMPLEXES

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1. Let $A$ be a finite abstract complex as defined by S. Lefschetz following A. W. Tucker, W. Mayer and J. W. Alexander. An open subcomplex $C$ of an abstract complex $D$ is a subset of $D$ (order, dimensions and incidences in $C$ determined by those in $D$) such that $x \in C$ and $y > x$ implies $y \in C$. It will be shown here that for every $A$ there is an open subcomplex $B$ of a simplicial complex such that the following homology groups using integer coefficients are isomorphic:

$$H^q(A) \cong H^q + \alpha(B), \quad q \text{ arbitrary},$$

where $\alpha = 0$ if $A$ has no elements of negative dimension and no zero-dimensional torsion coefficients and otherwise $\alpha > 0$. A result of Steenrod's shows that relation (1) then holds for any coefficient group. One of the principal uses of an abstract complex being to carry a homology theory, the present result shows that in this respect and for the finite case simplicially realizable complexes are as general as any abstract complexes.

2. Let $\| b_{ij} \|$ be the normal form of the $p, p - 1$ incidence matrix of $A$, and give each row of this matrix a name, $E^p_i$, and each column a name $E^{p-1}_j$. If this is done for each row and column of all the simultaneously reduced incidence matrices of $A$, the set $\{E\}$ may be made an abstract complex $C$ by defining as incidence relations $[E^p_i : E^{p-1}_j] = b_{ij}$. Obviously

$$H^q(A) \cong H^q(C), \quad q \text{ arbitrary}.$$  

Because the normal matrices are diagonal and $FF = 0$ in $C$, $[E^p_i : E^{p-1}_j] \neq 0$ implies that all other incidence relations involving either $E^p_i$ or $E^{p-1}_j$ are zero.

3. Suppose that for some $i$ $[E^p_i : E^{p-1}_j] = k \neq 0, 1, -1$. This cannot happen in a simplicial complex, so to make $C$ simplicial each such pair
$E_i^p, E_i^{p-1}$ must be replaced by a complex $D_i^p$ with the same homology groups and no incidence relations of absolute value greater than 1. The situation described in the last sentence of No. 2 makes it possible to do this without disturbing the rest of $C$.

I. The following adaptation to abstract complexes of "Subdivision by Section" replaces a complex $G$ by a complex $G'$ with the same homology groups: If $E^p \in G$ and $F E^p = C_1^p - 1 + C_2^p - 1$ where $C_i^p - 1$ are $(p - 1)$-chains of $G$, $E^p$ is replaced by $e_1^p, e_2^p, e_i^{p-1}$ to form $G'$, incidences being given by

$$F e_i^{p-1} = F C_i^{p-1}$$
$$F e_1^p = C_1^p - 1 + e_i^{p-1}$$
$$F e_2^p = C_2^p - 1 - e_i^{p-1}$$
$$[E^p + 1, e_i^p] = [E^p + 1, E^p], i = 1, 2$$

and all other incidences (not involving $E^p$) are the same in $G'$ as in $G$.

II. Without loss of generality $k$ may be assumed $> 1$ and the subscripts $i$ may be omitted from the $E$'s. Using I replace $E^p$ by $g_i^p, e_i^p, e_i^{p-1}$ so that $F e_i^{p-1} = 0, F e_i^p = E^p - 1 + e_i^{p-1}, F g_i^p = (k - 1) E^p - 1 - e_i^{p-1}, [E^p + 1, g_i^p] = [E^p + 1, e_i^p] = [E^p + 1, E^p] = 0$. Then unless $k - 1 = 1$ replace $g_i^p$ by $g_i^p, e_i^p, e_i^{p-1}$, etc., until after $n = k - 1$ steps

$$F e_i^{p-1} = F C_i^{p-1}$$
$$F e_i^p = E^p - 1 - e_i^{p-1} + e_i^{p-1}$$

where $e_i^{p-1} = e_i^{p-1} = 0$ and $e_i^p = g_i^p$, all other incidences involving these elements being zero. The situation is now as follows: $k$ new elements $e_i^p$ and $k - 1$ new elements $e_i^{p-1}$ have replaced $e_i^p$. Examination of formula 3 shows that the incidence $[E^p : E^p - 1] = k > 1$ has been replaced by several incidences all of absolute value 1; $e_i^p$ and $e_{i+1}^{p-1}$ have the two common faces $E^p - 1$ and $e_i^{p-1}$; $e_i^{p-1}$ is oriented to $e_i^p$ oppositely than to $e_{i+1}^{p-1}$, the last two of which circumstances are incompatible with simpliciality and impose the following changes:

III. Subdivide each $e_i^p$ twice more by means of I, finally getting the set \[ \{ s_i^p, t_i^p, u_i^p, e_i^{p-1} \} = D_i^p, i = 1, 2, \ldots, k \] with incidences given by

$$F e_i^{p-1} = F t_i^{p-1} = F s_i^{p-1} = 0$$
$$F u_i^p = -e_i^{p-1} + s_i^{p-1} + t_i^{p-1}$$
$$F t_i^p = -t_i^{p-1} - s_i^{p-1}$$
$$F s_i^p = E^p - 1 + s_i^{p-1}$$

all other incidences involving elements of $D_i^p$ being zero.

IV. Clearly $D_i^p$ is incidence-equivalent to an open 2-subcomplex $K^2$ of a simplicial complex where $s_i^p, t_i^p, u_i^p$ are represented by 2-simplexes and $e_i^{p-1}$,
$s_i^p - 1, t_i^p - 1$ by suitably chosen 1-simplexes on the boundaries of the 2-simplexes. $K^2$ has no vertices and some of the 1-simplexes on boundaries of 2-simplexes are missing so it is simplicially open.

V. In order to preserve dimension as far as possible, a technique for raising the dimension of $K^2$ is needed. If $\alpha \geq 0$ is an integer let $L^p + \alpha$ be the join $K^2 \bowtie \sigma$ where $\sigma$ is a new open $(p + \alpha - 3)$-simplex ($\sigma^{-1} = 1$). If now $s_i^p$ is replaced by $s_i^p \bowtie \sigma$ and similarly for all simplexes of $K^2$ the incidence formulas 4–7 remain unchanged, so $H_i^q + \alpha(L^p + \alpha)$ is isomorphic to the $q$th homology group of the complex $\{E_i^p, E_i^p - 1\}$.

4. The final complex $B$ is obtained as follows:

a. If the minimum dimension, $m$, of any element of $A$ is $\geq 0$ and $A$ has no zero-dimensional torsion coefficient, let $\alpha = 0$. If $m < 0$ and $A$ has no $m$-dimensional torsion coefficient, let $\alpha = -m$. If $m \leq 0$ and $A$ has an $m$-dimensional torsion coefficient let $\alpha = -m + 1$.

b. To each $D_i^p$ (corresponding to a pair $E_i^p, E_i^p - 1$ such as considered in No. 3) assign its $L_i^p + \alpha$. To each $E_i^p$ of $C$ with $FE_i^p = 0$ assign an open $(p + \alpha)$-simplex $W_i^p + \alpha$ such that neither $W$ nor its boundary $FW$ meets any previously assigned simplex. The set $B = \{L_i^p + \alpha, W_i^p + \alpha\}$ is an open complex of a simplicial complex and

$$H_i^q + \alpha(B) \approx H_i^q(A). \quad (1)$$

To see the latter notice that $B$ has the same structure (except for dimension if $\alpha > 0$) as $C$ with regard to all elements $E_i^p$ of $C$ which do not have the property $FE_i^p = \neq E_i^p - 1$. These $E_i^p$ are omitted from representation in $B$ because they have no effect on the homology groups of $C$. Hence by formula 2,

$$H_i^q + \alpha(B) \approx H_i^q(C) \approx H_i^q(A)$$

which gives formula 1.