test for exhaustiveness calls for reasoning so circuitous that it is omitted here. In any case, such a conclusion requires independent checking. These diagrams are of interest, as exhibiting schematic symmetries, or rotation periods 2 or 3 or 7. I think it correct to assert further (cf. §3) that the completion of each pattern by inclusion of its minor polyedra is uniquely determinate; so that 11 appears to be the total number of systems unlike in respect of numbers and contiguities.

Not inapposite may be an obiter dictum from the late mathematician of wide vision, Maxime Bocher. "Moreover, although the mathematical method is the traditional one for arriving at the truth concerning geometrical facts, it is not the only one. Direct appeal to the intuition is often a short and fairly safe cut to geometric results."


ON THE EXISTENCE OF MINIMAL SURFACES OF GENERAL CRITICAL TYPES

BY MARSTON MORSE AND C. TOMPKINS

INSTITUTE FOR ADVANCED STUDY AND PRINCETON UNIVERSITY

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We are concerned with minimal surfaces bounded by a simple closed curve $g$ lying in a euclidean space of $n$ dimensions $n > 1$. As is well known, the existence of a minimal surface of minimum type was first established by Douglas. In this connection one should refer to the significant work of Radó, McShane, Courant and others. The present paper is concerned with the existence of minimal surfaces of non-minimum as well as of minimum type. The surfaces considered are of the topological type of the circular disc. We have succeeded in making this study an application of the general theory of the critical points of functionals. That this is possible for ordinary regular positive definite problems of the calculus of variations has already been seen. Having in mind the possibility of applying the general theory to multiple integrals as well as to simple integrals, Morse has recently given an exposition [6] of the fundamental principles involved. The present paper has merely to verify the conditions laid down in the general theory. The two most difficult aspects appear in the proof of the upper-reducibility of the Douglas-Dirichlet functional and of the theorem that a homotopic critical point defines a minimal surface. Finally, an example is given in which there appear two minimal surfaces of minimum
type. By virtue of our theorems there then exists a minimal surface of non-minimum type.

The present paper is an abstract of results which will later appear in full.

1. The General Theory.—We shall here present only those concepts and theorems of the general theory which are essential to the theory of minimal surfaces.

We start with an abstract metric space $M$ with points $p, q, r, \ldots$. We make use of Vietoris chains and cycles on $M$ with coefficients in an arbitrary field. Let $F(p)$ be a real single-valued function of the point $p$ on $M$. We suppose that $F$ is finite and positive but in general not bounded. We define an $F$-deformation of $M$ as in Morse [6]. If $c$ is a finite constant, the subset of points of $M$ on which $F < c$ will be said to be below $c$. The subset of points of $M$ on which $F \leq c$ will be said to be below $c^+$.

We shall now describe three conditions on $F$ and $M$ which are useful in the sequel.

I. Bounded Compactness.—Under this condition the subset below $c^+$ is compact for each finite constant $c$.

II. Regularity at Infinity.—Corresponding to each compact subset $A$ of $M$, this condition requires the existence of a continuous deformation of $A$ such that the final image of $A$ is below some finite constant, and such that any subset of $A$ on which $F$ is bounded is deformed through a set of points on which $F$ is bounded.

III. Weak Upper-Reducibility at a Point $p$.—This condition is satisfied at $p$ if corresponding to each constant $c > F(p)$ there exists a neighborhood $N_c$ of $p$ relative to the set below $c^+$, and a deformation $\Delta$ of $N_c$ with the following properties:

(i) For some positive $\eta$ the points of $N_c$ not below $c - \eta$ are deformed with a displacement function onto a set below $(c - \eta)^+$.

(ii) The points of $N_c$ below $c - \eta$ remain below $c - \eta$.

Homotopic critical points and homotopic ordinary points, $k$-caps, $a$-homologies, critical sets and their type numbers are defined as in Morse [6].

The simplest general theorem is as follows:

**Theorem 1.** Let $F$ be a function on $M$ satisfying conditions I, II, III and possessing homotopic critical points which are finite in number. Let $R_k$ be the $k$th connectivity number of $M$ and $M_k$ the sum of the $k$th type numbers of the homotopic critical points of $M$. If $M_k$ and $R_k$ are finite and if we set $e_k = M_k - R_k$ the following relations hold

\[
\begin{align*}
    e_0 &\geq 0, \\
    e_1 - e_0 &\geq 0, \\
    \ldots
\end{align*}
\]

\[e_n - e_{n-1} + \ldots + (-1)^n e_0 = 0\]

where $n$ is the maximum of the indices of the numbers $M_k$. 

The condition that the homotopic critical points be finite in number will be removed in three different types of theorems. If one assumes merely that the number of homotopic critical points below each finite constant \( c \) is finite, and that the numbers \( M_k \) and \( R_k \) are finite, the relations (1) again hold with the equality deleted and \( n \) in general infinite. The least restrictive theorem of all involves the notion of a relative homology class defined as follows.

Let \( u \) be a \( k \)-cycle mod \( F \leq a \), non-bounding below \( c^+ \) mod \( F \leq a \); \( c > a \). Let \( z \) be the homology class containing \( u \) taken below \( c^+ \) and mod \( F \). A constant \( b \) such that there is a cycle of \( z \) below \( b^+ \) will be termed a cycle bound of \( z \).

A constant \( b \) such that there is a cycle of \( z \) below \( b \) will be termed a cycle bound of \( z \).

We have the following theorem:

**Theorem 2.** The cycle bounds of the relative homology class \( z \) have a minimum, and this minimum is assumed by \( F \) in at least one homotopic critical point.

A corollary of this theorem is that the existence of at least two disconnected critical sets of minimal surfaces of minimum type implies the existence of at least one minimal surface of non-minimum type.

More generally the relations (1) can be replaced by group relations giving a decomposition of a maximal group of caps. We defer this exposition until the paper is published in full. No finiteness conditions are assumed in this general theory. Counting of type numbers is replaced by comparisons involving isomorphisms of a special type.

3. **Minimal Surface Theory.**—We shall assume that our curve \( g \) is represented in the form \( x_i = g_i(s) \) where \( s \) is the arc length on \( g \). We suppose further that \( g_i(s) \) has the period \( 2\pi \) in \( s \) and that for each \( i, g_i(s) \) satisfies a Lipschitz condition. We shall admit other representations of \( g \) of the form

\[
x_i = g_i[\varphi(\alpha)] = \varphi_i(\alpha)
\]

where \( \varphi(\alpha) \) is a continuous non-decreasing function of \( \alpha \) such that \( \varphi(\alpha + 2\pi) = \varphi(\alpha) + 2\pi \). Let \( (r, \theta) \) be polar coordinates in a plane of rectangular coordinates \((u, v)\). Corresponding to the curve \([p]\) there exists a harmonic surface \( S \) of the form \( x_i = x_i(r, \theta) \) defined over the disc \( r \leq 1 \) and such that \( x_i(1, \theta) = \varphi_i(\theta) \). With Douglas we set

\[
A(\varphi) = \frac{1}{16\pi} \int_0^{2\pi} \int_0^{2\pi} \sum_i \frac{[\varphi_i(\alpha) - \varphi_i(\beta)]^2}{\sin^2 \frac{\alpha - \beta}{2}} \, d\alpha \, d\beta
\]

where \( Q \) represents the square \( 0 \leq \alpha \leq 2\pi, 0 \leq \beta \leq 2\pi \).

Under the three point condition on \( \varphi(\alpha) \) we understand that \( \alpha = \varphi(\alpha) \) for three given distinct values of \( \alpha \) on the interval \( 0 \leq \alpha < 2\pi \). Let \( \Omega \) be the space of those transformations \( \varphi(\alpha) \) which satisfy the three point condi-
tion and for which \( A(\varphi) \) is finite. We term \( \varphi \) a point on \( \Omega \). To two points \( \varphi \) and \( \psi \) of \( \Omega \) we assign a distance

\[
\varphi \psi = \max | \varphi(\alpha) - \psi(\alpha) | , \quad 0 \leq \alpha \leq 2\pi.
\]

As is well known the subset of points \( \varphi \) of \( \Omega \) for which \( A(\varphi) \) is at most a finite constant is compact. Thus if \( M = \Omega \) and \( F = A(\varphi) \) the condition I of §1 is satisfied.

We come to condition II of §1. Corresponding to each point \( \varphi \) of \( \Omega \) we shall define a continuous deformation \( \Delta_\varphi \) of \( \Omega \). In this deformation the time \( t \) shall run from 0 to 1 inclusive, and an arbitrary point \( \psi \) on \( \Omega \) shall be replaced at the time \( t \) by the point

\[
\psi'(\alpha) = t\varphi(\alpha) + (1 - t)\psi(\alpha).
\]

We introduce the integral

\[
H(\psi) = \frac{1}{16\pi} \int_0^1 \int_0^1 \left| \frac{\psi(\alpha) - \psi(\beta)}{\sin \frac{\alpha - \beta}{2}} \right|^2 \ d\alpha \ d\beta
\]

and show that a necessary and sufficient condition that \( A(\psi) \) be finite is that \( H(\psi) \) be finite. It then follows readily that the space \( \Omega \) is continuously deformable on itself into a point and is regular at infinity.

To establish the weak upper-reducibility of \( A(\psi) \) at \( \varphi \), let \( Q_\epsilon \) be the subset of \( Q \) on which \( |\alpha - \beta| < \epsilon \text{ mod } 2\pi \). We then set

\[
A(\psi) = A_\epsilon(\psi) + A^*_\epsilon(\psi)
\]

\[
H(\psi) = H_\epsilon(\psi) + H^*_\epsilon(\psi)
\]

where \( A_\epsilon(\psi) \) and \( H_\epsilon(\psi) \) are obtained respectively from the integrals \( A(\psi) \) and \( H(\psi) \) by replacing the domain of integration \( Q \) by \( Q_\epsilon \). Referring to the deformation \( \psi'(\alpha) \) of \( \psi(\alpha) \) we set

\[
A(\psi') = a(t) \quad A_\epsilon(\psi') = a(t, \epsilon) \quad A^*_\epsilon(\psi') = a^*(t, \epsilon)
\]

\[
H(\psi') = h(t) \quad H_\epsilon(\psi') = h(t, \epsilon) \quad H^*_\epsilon(\psi') = h^*(t, \epsilon).
\]

We then show that for a fixed \( \epsilon \) and bounded \( A(\psi) \) the functions

\[
a^*_\epsilon(t, \epsilon) \quad a(t) - h(t, \epsilon)
\]

tend to zero uniformly with respect to \( t \) as \( \psi \) tends to \( \varphi \). We also need the fact that for points \( \psi \) on a set of \( \Omega \) for which \( A(\psi) \) is bounded the ratio of \( A_\epsilon(\psi) \) to \( H_\epsilon(\psi) \) tends uniformly to 1 as \( \epsilon \) tends to 0. Upon noting the effect of \( \Delta_\varphi \) on \( H(\psi) \) and making use of the way in which the various functionals derived from \( H(\psi) \) approximate the corresponding functionals derived from \( A(\psi) \), we arrive at a proof of the weak upper-reducibility of \( A(\psi) \) at \( \varphi \).

A transformation \( \varphi(\alpha) \) of \( \Omega \) which defines a minimal surface will be termed
differentially critical. We need to show that a homotopic critical point $\varphi$ is a differential critical point. To that end let $D(q)$ denote the Dirichlet integral defined by the representation $q_i(\alpha)$ of $g$. Suppose $D(q)$ is finite and let $\lambda(\alpha)$ be a function of $\alpha$ of class $C'''$ with period $2\pi$. If $\epsilon_0$ is a sufficiently small positive constant and $|\epsilon| \leq \epsilon_0$, the transformation $\mu = \alpha + \epsilon \lambda(\alpha)$ from $\alpha$ to $\mu$ has a single-valued inverse, $\alpha = m(\mu, \epsilon)$ and the Dirichlet integral $d(q, \epsilon)$ defined by the boundary curve

$$
x_i = q_i[m(\theta, \epsilon)], \quad |\epsilon| < \epsilon_0
$$

has the form

$$
d(q, \epsilon) = D(q) + \epsilon S(q) + \epsilon^2 R(q, \epsilon).
$$

Here $S(q)$ is an integral depending continuously on $q$, while $R(q, \epsilon)$ and $R_\epsilon(q, \epsilon)$ are in absolute value at most a constant multiple of $D(q)$. Moreover the condition $S(q) = 0$ implies that the harmonic surface defined by $q$ is minimal. We can regard the transformation $\mu = \alpha + \epsilon \lambda(\alpha)$ as defining a deformation of boundary values, $\epsilon$ being the time. Making use of (3.2), it is then easy to show that a homotopic critical point is a differential critical point. Naturally the deformations which we have used above do not lead to transformations $\psi(\alpha)$ which satisfy the three point condition, but upon making suitable conformal transformations of the disc $r \leq 1$, the three point condition is satisfied and all is well.

All the conditions of the theory of $\S 1$ are fulfilled and the desired results about minimal surfaces can be read from the general theory.

4. The Example.—This example will be given in the space of coordinates $x, y, z$. Consider the closed curve composed of the following four arcs taken successively:

$$
\begin{align*}
x &= e \\
y &= e \\
x &= -e \\
y &= -e
\end{align*}
$$

where $e$ and $a$ are positive constants with $a^2 = 1 - e^2$. Let $C_\epsilon$ be a closed curve obtained by rounding off the corners of the above closed curve. If $e$ is sufficiently small and the rounding off process is suitably made, it is easy to see that there exist two minimal surfaces of minimum type bounded by $C(\epsilon)$. Moreover, these two surfaces do not define points of $\Omega$ in the same connected critical set. It follows from our theory that $C(\epsilon)$ bounds at least one minimal surface of non-minimum type.


ADDITIVE SET FUNCTIONS ON GROUPS

BY S. BOCHNER

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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1. We consider on an arbitrary set $G$ a family $X$ of subsets $\{E\}$ with the properties

1) $X$ contains the empty set $0$; if $E \in X$, then $G = E \in X$
2) if $E_1, E_2 \in X$, then $E_1 \cdot E_2 \in X$ and $E_1 + E_2 \in X$,

and a Jordan volume $vE$ with the properties

3) $0 \leq vE \leq 1$, $v0 = 0$, $vG = 1$
4) if $E_1 \cdot E_2 = 0$, then $v(E_1 + E_2) = vE_1 + vE_2$.

We do not assume that $X$ is completely additive, or that $vE$ is completely additive relative to $X$. The class of point functions $f(x)$ which are Riemann integrable relative to $vE$ will be denoted by $R$, the integral of $f(x)$ over any set $E$ will be denoted by $\int_E f(x) \, dv$ and the (incomplete) Banach space of functions $f(x)$ with the norm

$$||f||_p = \left( \int_G |f(x)|^p \, dv \right)^{1/p}$$

will be denoted by $R_p$, $p \geq 1$.

2. Many properties of (finitely) additive set functions of bounded variation which are usually proven under assumptions of complete additivity can be established for our general case. It can be shown that in the Banach space $AC$ of absolutely continuous set functions $F(E)$ with the variation as norm, the finitely valued functions are dense. In other words, the space $AC$ is not greater than, but equal to, the Banach closure of the space $R_1$. The same is true of the Banach spaces $V_p$, $p > 1$, of the set functions $F(E)$ with the norm

$$||F||_p = \sup \left( \sum_{n=1}^N |F(E_n)|^p \cdot |vE|^{1-p} \right)^{1/p}.$$