The importance of the present procedure is that it provides a straightforward method of finding the kernel that corresponds to a given differential operator. As other examples we note that if \((A)\) is altered by omitting\(^4\) the factor \((1 + xD)\) in \((3)\) the kernel becomes \(t(x + t)^{-2}\); if in addition the factor \(1 + 2^{-1}xD\) is omitted the kernel becomes \(t^2(x + t)^{-3}\), etc. If the system \((A)\) is altered by replacing \(L[f(x)]\) by its iterate \(L^2[f(x)]\) we find that the Green's function for the corresponding truncated system is

\[
\int_0^\infty G_k(x, y)G_k(y, t)dy
\]

and that the Green's function for the system of infinite order is \([\log(x/t)]\) \([x - t]^{-1}\), as one would expect from earlier consideration of R. P. Boas\(^6\) and the author.

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**THEOREMS IN THE INVERSE PROBLEM IN THE CALCULUS OF VARIATIONS**

**BY JESSE DOUGLAS**

**BROOKLYN, N. Y.**

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1. **Introduction.**—In a recent issue of these PROCEEDINGS the author has announced and presented the essential features of a solution of the inverse problem of the calculus of variations recently found by him.\(^1\) This problem is: Given a curve family, \(y_i'' = F_i(x, y_j, y_j'), (i, j = 1, \ldots, n)\), in \((n + 1)\)-dimensional space; to find, if existent, a variation problem, \(\int \varphi(x, y_j, y_j')dx = \min\), having this curve family as the totality of its extremals.

A fully detailed account of our solution will appear in one of the mathematical journals. The present note summarizes the results of this detailed paper, which it is here our purpose to state.

We have given in our forthcoming paper a general method applying to an \((n + 1)\)-dimensional space, and then carried out this plan completely
for the most important and interesting case of 3 dimensions. This results in a classification of all families \( \mathfrak{F} \) of \( \infty^4 \) curves in \( xyz \)-space:

\[
y'' = F(x, y, z, y', z'), \quad z'' = G(x, y, z, y', z'), \quad (1.1)
\]

into extremal and non-extremal, together with a determination in the former case of the degree of generality, i.e., the number of arbitrary functions and constants, involved in the corresponding variation problem

\[
\int \varphi(x, y, z, y', z')dx = \min. \quad (1.2)
\]

Illustrative examples for all the more important cases are given at the end of this note. These are more completely described in our detailed paper.

In stating our results, the following notation introduced by E. Kasner in his review of Riquier's treatise on differential systems is found useful:

\[
\infty^{m_1f(n_1)} + \ldots + m_kf(n_k)
\]

(1.3)

denotes an infinitude involving \( m_1 \) arbitrary functions of \( n_1 \) arguments, \ldots, \( m_k \) arbitrary functions of \( n_k \) arguments. This extends the classic notation \( \infty^m \) for an infinitude involving \( m \) arbitrary constants.

All functions occurring in our work are supposed to be analytic.

2. The Matrix \( \Delta \).—We define the following symbols, functions of \( x, y, z, y', z' \), whose values are known when the curve family \( \mathfrak{F} \) or (1.1) is given.

\[
A = \frac{d}{dx}F_y - 2F_z - \frac{1}{2}F_{y'}(F_y' + G_z),
\]

\[
B = -\frac{d}{dx}F_y' + \frac{d}{dx}G_z + 2(F_y - G_z) + \frac{1}{2}(F_y' - G_z')(F_y' + G_z),
\]

\[
C = -\frac{d}{dx}G_y' + 2G_y + \frac{1}{2}G_y'(F_y' + G_z').
\]

(2.1)

\[
\frac{d}{dx} \equiv \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z} + F\frac{\partial}{\partial y'} + G\frac{\partial}{\partial z'},
\]

(2.2)

representing total differentiation as to \( x \) along an arbitrary curve of the given family (1.1).

From \( A, B, C \) we derive \( A_1, B_1, C_1 \) by the following formulas:
\[ A_1 = \frac{dA}{dx} - F_y^A - \frac{1}{2} F_z^B, \]
\[ B_1 = \frac{dB}{dx} - G_y^A - \frac{1}{2} (F_y^A + G_z^A)B - F_z^C, \]
\[ C_1 = \frac{dC}{dx} - \frac{1}{2} G_y^B - G_z^C; \]
\[ \text{(2.3)} \]

while \( A_2, B_2, C_2 \) are derived from \( A_1, B_1, C_1 \) by the same formulas.

Our results depend to a large extent on the rank of the matrix

\[ \Delta = \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}. \]

\[ \text{(2.4)} \]

For instance, it is at least a necessary condition for an extremal family that the determinant of this matrix be equal to zero. By non-satisfaction of this condition, therefore, examples of non-extremal families can be constructed at pleasure.

We begin, consequently, with a classification into cases according to the rank of \( \Delta \), which will be followed by the appropriate sub-classifications in the statement of our theorems.

**Case I:** \( \begin{vmatrix} A & B & C \end{vmatrix} = 0; \) i.e., \( A = 0, B = 0, C = 0. \)

**Case II:** \( \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \end{vmatrix} = 0, \begin{vmatrix} A & B & C \end{vmatrix} \neq 0. \)

**Case III:** \( \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \end{vmatrix} = 0, \begin{vmatrix} A & B & C \\ A_2 & B_2 & C_2 \end{vmatrix} \neq 0. \)

**Case IV:** \( \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \end{vmatrix} \neq 0. \)

Here, in writing a matrix \( = 0 \) we mean that each determinant resulting from it by the suppression of columns only is equal to zero, and \( \neq 0 \) means that at least one such determinant is not equal to zero.

By the recursion formulas (2.3) it is seen that the cases thus described are precisely those of rank 0, 1, 2, 3 of the matrix \( \Delta \), respectively.

3. **The Fundamental Differential System \( S \).**—As we prove in our main paper, the solution of the inverse problem of the calculus of variations for the given curve family (1.1) is equivalent exactly to the solution of the following linear differential system, \( S \), for the unknown functions \( L, M, N \) of \( x, y, z, y', z' \):
\[
\begin{align*}
\frac{dL}{dx} + F_y' L + G_y' M &= 0, \\
\frac{dM}{dx} + \frac{1}{2} F_z' L + \frac{1}{2} (F_y' + G_y') M + \frac{1}{2} G_y' N &= 0, \\
\frac{dN}{dx} + F_z' M + G_z' N &= 0; \\
AL + BM + CN &= 0; \\
L_{y'} = M_{y'}, N_{y'} &= M_{z'}; \\
LN - M^2 &\neq 0. 
\end{align*}
\]

(3.1)

4. The "Critical Cone."—The inequation (3.1), last relation of the system \( S \), is very important. Its negation,

\[
LN - M^2 = 0, 
\]

(4.1)
defines a quadric cone in an auxiliary \( LMN \)-space, whose significance first appears in our work on the inverse problem, and which we call the "critical cone," denoting it by \( \mathcal{K} \).

The purely algebraic equation (3.14) of the differential system, \( S \):

\[
AL + BM + CN = 0,
\]

(4.2)
defines a plane \( \mathfrak{B} \) in the \( LMN \)-space passing through the vertex of the critical cone, located at the origin. In the discussion of Case II—which is the most interesting and varied in its results—much depends on whether the plane \( \mathfrak{B} \) intersects the critical cone \( \mathcal{K} \) in two distinct generators (real or conjugate imaginary) or, on the other hand, is tangent to this cone.

The intersection of \( \mathfrak{B} \) with \( \mathcal{K} \) is determined by the quadratic equation

\[
A\xi^2 + B\xi + C = 0,
\]

(4.3)
whose roots, known functions of \( x, y, z, y', z' \), will be denoted by \( \lambda, \mu \). Accordingly, we make the following subdivision of Case II:

Case IIa: \( B^2 - 4AC \neq 0, \) or \( \lambda \neq \mu \);

Case IIb: \( B^2 - 4AC = 0, \) or \( \lambda = \mu \).

In Case IIa the following symbols intervene in the statement of our results, where the importance of the hypothesis \( \lambda \neq \mu \) is seen in the presence of \( \lambda - \mu \) in various denominators.
\[
\alpha = \frac{\lambda \lambda_\alpha - \lambda_\gamma'}{\lambda - \mu}, \quad \beta = \frac{\mu \mu_\alpha - \mu_\gamma'}{\lambda - \mu},
\]
\[
H = \frac{1}{2} F_\alpha \lambda^2 - \frac{1}{2} (F_\alpha' - G_\alpha) \lambda - \frac{1}{2} G_\alpha',
\]
\[
I = \lambda_\alpha + H_\alpha' - R,
\]
\[
J = \frac{(H - P) \beta + K}{\lambda - \mu},
\]
\[
K = \mu \mu_\alpha - \mu_\gamma + \mu_\alpha' + \mu \mu_\gamma' - \mu_\alpha',
\]
\[
P = \frac{1}{2} F_\alpha \mu^2 - \frac{1}{2} (F_\alpha' - G_\alpha') \mu - \frac{1}{2} G_\alpha',
\]
\[
Q = \mu_\alpha + P_\alpha' - J,
\]
\[
R = \frac{-(H - P) \alpha + S}{\lambda - \mu},
\]
\[
S = \lambda \lambda_\alpha - \lambda_\gamma + H \lambda_\alpha' - \lambda_\gamma H_\alpha' + H_\gamma'.
\]  

(4.4)

In Case IIb the following symbols are important:

(I) \[= \frac{1}{2} F_\alpha \lambda^2 - \frac{1}{2} (F_\alpha' - G_\alpha') \lambda - \frac{1}{2} G_\alpha',\]

(II) \[= F_\alpha \lambda - \frac{1}{2} (F_\alpha' - G_\alpha'),\]

(III) \[= \lambda_\alpha + (I)_\alpha,\]

(IV) \[= (II)_\alpha,\]

(V) \[= \lambda \lambda_\alpha - \lambda_\gamma,\]

(VI) \[= \lambda \lambda_\alpha - \lambda_\gamma + (I)_\alpha' + \lambda(I)_\alpha' + (I)_\alpha' + (II)(V),\]

(VII) \[= 2 \lambda_\alpha + 2 \lambda_\alpha' (II) - \lambda (II)_\alpha' + (II)_\gamma',\]

(VIII) \[= \lambda (V I)_\alpha' - (V I)_\alpha' + (V)_\gamma' - \lambda (V)_\alpha + \lambda_\alpha'(V I) - (I)(V)_\alpha' + (III)(V) - (V)(V I),\]

(IX) \[= \lambda (V I I)_\alpha' - (V I I)_\alpha' - 2 \lambda_\alpha' (I) + 2 \lambda_\alpha' - 2 \lambda_\alpha' + (IV)(V).\]  

(4.5)

Also, we denote by \((V I)'\), \((V I X)'\) the expressions (VI), (IX) without their respective last terms.

5. Statement of Results.—Theorem I. Every curve family \(\xi\) which obeys the conditions \(A = 0, B = 0, C = 0\) of Case I is an extremal family, and the
generality of the corresponding variation problem is expressed by the symbol $\alpha_{2/3} + 2\alpha_{2/2}$.

It should be stated here that, in counting the arbitrary functions involved in the determination of $\varphi$ when $F, G$ are given, we omit the function $\nu(x, y, z)$ in the arbitrary exact differential $d\nu(x, y, z)$ that may always be added to $\varphi dx$ without changing the extremals.

THEOREM II. In Case IIa, if the conditions

\begin{align*}
\alpha = 0, \beta = 0, S = 0, K = 0
\end{align*}

are verified, then the curve family $\mathfrak{I}$ is one of extremals, and the generality of the corresponding variation problem is $\propto 2\alpha_{2/2}$.

THEOREM III. In Case IIa, if the conditions (5.1) are not verified, but certain other conditions (stated in our forthcoming complete paper) are, then the given curve family $\mathfrak{I}$ is of extremal nature and belongs to $\propto 1/2 + 1/1$ different variation problems.

THEOREM IV. In Case IIb, if the conditions

\begin{align*}
(V) = 0, (VI') = 0, (IX') = 0
\end{align*}

are satisfied, then the given curve family $\mathfrak{I}$ can be identified with the extremals of a class of variation problems whose generality is $\propto 2\alpha_{2/2}$.

As a preliminary to the statement of our next theorems, we introduce symbols to represent the second order determinants contained in the two-by-three matrix whose non-vanishing figures in Case III, namely:

\begin{align*}
\Delta_1 = BC_1 - B_1 C, \Delta_2 = CA_1 - C_1 A, \Delta_3 = AB_1 - A_1 B.
\end{align*}

Also, we let

\begin{align*}
D = \Delta_1 \Delta_3 - \Delta_2^2.
\end{align*}

Making then in the differential system, $S$, or (3.1), the substitutions

\begin{align*}
L = \rho \Delta_1, M = \rho \Delta_2, N = \rho \Delta_3,
\end{align*}

we obtain a differential system of the following simple form for $\rho(x, y, z, y', z')$ as unknown function:

\begin{align*}
\rho_x = E_1 \rho, \rho_y = E_2 \rho, \rho_z = E_3 \rho, \rho_{y'} = E_{4\rho}, \rho_{z'} = E_{6\rho}, \rho \neq 0; \quad (5.6)
\end{align*}

where the $E_i$ are rational expressions in the partial derivatives of $F, G$ involving only $D$ as denominator, and therefore existing as calculable known functions if $D \neq 0$.

THEOREM V. In Case III, if $D = 0$, then the given curve family is non-extremal.

THEOREM VI. In Case III, if $D \neq 0$, then a necessary and sufficient condition for the given curve family $\mathfrak{I}$ to be one of extremals is that the differential

\begin{align*}
E_1 dx + E_2 dy + E_3 dz + E_4 dy' + E_6 dz'
\end{align*}

(5.7)
be exact. The corresponding variation problem is essentially uniquely determined, i.e., up to a constant multiplier and an arbitrary additive exact differential, \( dv(x, y, z) \).

**THEOREM VII.** All curve families \( \mathfrak{F} \) in Case IV are non-extremal.

The preceding theorems cover all the more interesting and important cases. Certain additional minor cases are discussed in our detailed paper.

6. **Examples.**—THEOREM I: \( y'' = f(z'), z'' = 0 \), where \( f \) denotes an arbitrary function. This includes the case of the straight lines: \( y'' = 0, z'' = 0 \) treated by G. Hamel in a well-known paper.\(^4\)

**THEOREM II:** The "separated case:" \( y'' = F(x, y, y'), z'' = G(x, z, z'), B \neq 0 \).

**THEOREM III:** The \( \infty^4 \) catenaries which lie in planes perpendicular to the \( xz \)-plane and the director of each of which coincides with the trace of its plane upon the \( xz \)-plane:

\[
y'' = \frac{1 + y'^2 + z'^2}{y}, \quad z'' = 0.
\]

**THEOREM IV:** \( y'' = z, z'' = 0 \).

**THEOREM V:** \( y'' = y^2 + z^2, z'' = 0 \).

**THEOREM VI:** \( y'' = z^2, z'' = y^2 \). The corresponding variation problem is: \( \int (y'z' + \frac{1}{3}y^3 + \frac{1}{3}z^3) \, dx = \min. \), up to the slight possibility of modification expressed in Theorem VI.

**THEOREM VII:** \( y'' = y^2 + z^2, z'' = y \).

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3. Subscripts attached to the roman numerals in parentheses denote partial differentiation.