CONCERNING INTERSECTING CONTINUA

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Communicated November 2, 1942

In this paper a study is made of certain relationships involving (1) a
continuum $M$, (2) its boundary $\beta$, (3) the components of $M - \beta$, (4) the
components of the closures of the components of $M - \beta$ and (5) the common part of $M$ and another continuum $K$. Numbered axioms and chapters referred to are axioms and chapters of the
author's book, "Foundations of Point Set Theory."

Theorem 1. If, in a space satisfying Axioms 0 and 1, $M$ is a compact
continuum, with a boundary $\beta$ such that, if $D$ is a component of $M - \beta$, $\beta \cdot D$ is connected, and $K$ is a continuum intersecting $\beta$, then $M \cdot K$ is a con-
tinuum.

Proof. By Theorem 3 of my paper, "Concerning a Continuum and Its
Boundary," $\beta$ is connected. Hence so is $K + \beta$. If $M \cdot K$ is $K$ or a
subset of $\beta$ then $M \cdot K + \beta$ is $K + \beta$ or $\beta$. If $M \cdot K$ is neither $K$ nor a
subset of $\beta$ then $(K + \beta) - \beta$ is the sum of two mutually separated point
sets $M \cdot K - M \cdot K \cdot \beta$ and $K - M \cdot K$. Hence $(M \cdot K - M \cdot K \cdot \beta) + \beta$
is connected. But this point set is identical with $M \cdot K + \beta$.

Theorem 1 does not remain true if, in the statement of its hypothesis,
the stipulation that $M$ is compact is replaced by the stipulation that $K$ is
compact even though there is added, to this hypothesis, the additional
stipulation that the entire boundary of every component of $M - \beta$ is a
connected subset of $\beta$ and the conclusion is so weakened as to require only
that there exists a continuum containing $M \cdot K$ and lying in $M \cdot K + \beta$.
This may be seen with the help of Example 2 of C. B. However if, in the
above proof of Theorem 1, Theorem 3 of C. B. is replaced by Theorem 4
of that paper, the resulting argument establishes the following result.

Theorem 2. Theorem 1 remains true if, in the statement of its hypothesis,
the requirement that $M$ be compact is replaced by the requirement that Axiom
2 hold true.

Theorem 3. If, in a space satisfying Axioms 0 and 1, the continuum $K$
contains $\beta$, the boundary of the compact continuum $M$ and, for every com-
ponent $D$ of $M - \beta$, $K \cdot D$ is a continuum, then $M \cdot K$ is a continuum.

Theorem 4. Theorem 3 remains true if the requirement that $M$ be compact
is replaced by the requirement that space satisfy Axiom 2.

Theorems 3 and 4 may be easily established with the aid of Theorems 1
and 2, respectively, of C. B. Theorem 3 may also be easily proved with
the help of Theorem 9 of my paper, "Concerning Accessibility."
It may be seen from Example 2 of C. B. that Theorem 3 does not remain true if $K$ instead of $M$ is required to be compact.

**Theorem 5.** If, in a space satisfying Axioms 0 and 1, $M$ is a continuum, $K$ is a compact continuum intersecting $M$ and, for every component $E$ of $S - M$ that intersects $K$, $T_E$ is a connected point set intersecting $M - K$ such that, for every component $Q$ of $K - E$, whose closure intersects $M$, $T_E$ intersects every component of $\overline{Q} \cdot M$ and $T$ is the sum of the point sets $T_E$ for all $E$’s then $M \cdot K + T$ is connected.

**Proof.** Suppose $M \cdot K + T$ is the sum of two mutually separated point sets $F_1$ and $F_2$. Every component of $T$ intersects $M - K$ and lies either in $F_1$ or in $F_2$. Hence $F_1 \cdot M - K$ and $F_2 \cdot M - K$ exist. Since these point sets are mutually separated and their sum is the closed set $M - K$, therefore they are closed. Hence $K - M - K$ contains a connected point set $L$ such that $L$ intersects $F_1$ and $F_2$. Let $W$ denote the component of $S - M$ which contains $L$. The point set $T_W$ is a connected subset of $T$ intersecting both $F_1$ and $F_2$. This involves a contradiction.

Theorem 5 does not remain true if the stipulation that $K$ is compact is replaced by the stipulation that $M$ is compact. Consider the example obtained by interchanging $M$ and $K$ in Example 2 of C. B. But the following theorem holds.

**Theorem 6.** Theorem 5 remains true if the requirement that $K$ be compact is replaced by the requirement that space satisfy Axiom 2.

**Proof.** Suppose $M \cdot K + T$ is the sum of the mutually separated point sets $F_1$ and $F_2$. By a modification of Theorem 6 of my paper, “Concerning Domains Whose Boundaries Are Compact,” there exists a component $W$ of $S - M$, intersecting $K$, whose boundary intersects each of the closed point sets $F_1 \cdot M - K$ and $F_2 \cdot M - K$. As in the above proof of Theorem 5, this leads to a contradiction.

The following theorem easily follows with the help of Theorem 6.

**Theorem 7.** If, in a space satisfying Axioms 0, 1 and 2, the continuum $K$ contains the boundary of the continuum $M$ and every component of $S - M$ has a connected boundary then $M \cdot K$ is connected.

The following theorem may be easily established with the help of Theorem 5.

**Theorem 8.** If, in a space satisfying Axioms 0 and 1, $M$ is a continuum and $K$ is a compact continuum intersecting $M$ and every component of $S - M$ that intersects $K$ has a connected boundary then $M \cdot K$ plus the boundaries of all components of $S - M$ that intersect $K$ is connected.

Theorem 8 does not remain true if its conclusion is replaced by “$M \cdot K$ is a subset of a component of $M \cdot K$ plus the boundary of $M$.” Indeed it does not remain true if its hypothesis is strengthened by the addition of the stipulation that the boundary of every component of $S - M$ is connected, and its conclusion is replaced by “$M \cdot K$ is a subset of some compact subcontinuum of $M$.” Consider the following example.
EXAMPLE 1. In a Cartesian plane $E$ let $T$ denote a totally disconnected compact and perfect point set lying on $OY$ and such that the ordinate of its highest point is $-10$. Let $A$ and $B$ denote the points $(-10, 0)$ and $(10, 0)$, respectively. Let $H$ denote the set of all points of the graph of $y = \sin \left(\frac{1}{x}\right)$ whose abscissae are neither less than $-1/\pi$ nor greater than $1/\pi$. Let $N$ denote the set of all points of $OX$ that lie between $A$ and $(-1/\pi, 0)$ or between $B$ and $(1/\pi, 0)$. Let $M$ denote the point set $O + H + N + A + B$ and let $K$ denote the sum of all straight line intervals with one end-point at one of the points $A$ and $B$ and the other one at a point of $T$. Let $S$ denote $M + K$. Let $\Sigma$ denote the subspace of $E$ whose points are the points of $S$. Here $\beta$ consists of two points, $A$ and $B$.

However, the following theorem holds true.

Theorem 9. If, in a space satisfying Axioms 0 and 1, $\beta$ is the boundary of the continuum $M$ and $K$ is a compact continuum intersecting $M$, and the common part of $M$ and the boundary of each component of $S - M$ that intersects $K$ is connected, then $M \setminus K$ is a subset of a component of $M \setminus K + \beta$.

Theorem 10. If, in a space satisfying Axioms 0 and 1, $K$ is a compact continuum intersecting the continuum $M$ and, for every component $E$ of $S - M$ that intersects $K$, $K \cdot E \cdot M$ is a subset of some component of $M \cdot K$ then $M \cdot K$ is connected.

Proof. For each component $E$ of $S - M$ that intersects $K$, let $T_E$ denote the component of $M \cdot K$ that contains $K \cdot E \cdot M$ and let $T$ denote the sum of all $T_E$'s. By Theorem 5, $M \cdot K + T$ is connected. But it is identical with $M \cdot K$.

It may be seen from the following example that Theorem 10 does not remain true if the requirement that $K$ be compact is omitted even though the resulting hypothesis is strengthened by the addition of the stipulation that (1) $K$ contains $\beta$ and (2) every component of $S - M$ is bounded by a connected subset of $M$ and so is every component of $M - \beta$ and, furthermore, if $D$ is a component of $M - \beta$, every component of $S - D$ is so bounded.

EXAMPLE 2. In a Cartesian plane $E$, let $A$ and $D$ denote the points $(0, 2)$ and $(0, 1)$ and, for each $n$, let $A_n$ and $B_n$ denote the points $(1/n, 0)$ and $(-1/n, 0)$, respectively, and let $AA_n$ and $AB_n$ denote straight line intervals with end-points as indicated. Let $M$ and $K$ denote $D + AB_1 + AB_2 + \ldots$ and $D + AA_1 + AA_2 + \ldots$, respectively. Let $S$ denote $M + K$. Let $\Sigma$ denote the subspace of $E$ whose points are the points of $S$.

Theorem 10 does not remain true if $M$ instead of $K$ is required to be compact. To see this, interchange $M$ and $K$ in the description of Example 2 of C. B.

Of Theorems 5–10, Theorem 7 is the only one that remains true if, in its statement, $S - M$ is replaced by $M - \beta$.

The following theorem may be easily proved.
**Theorem 11.** If, in a space satisfying Axiom 0, the closed point set $M$ intersects the connected point set $K$ and $M - K$ is the sum of two mutually separated point sets $H$ and $L$ then both $H$ and $L$ intersect the boundary of $M$.

**Theorem 12.** If, in a space satisfying Axioms 0 and 1, the compact point set $\beta$ is the boundary of the compactly connected continuum $M$ and, for every component $D$ of $M - K$ and component $E$ of $S - D$, the boundary of $E$ is a subset of a component of $\beta$ then if $K$ is a compact continuum intersecting $M$, $M - K$ is a subset of a component of $M - K + \beta$.

**Proof.** Let $N$ denote the point set obtained by adding together all components of $M - K + \beta$ that intersect $M K$. Since $M - K$ and $\beta$ are closed and compact, $N$ is closed. Suppose it is the sum of two mutually exclusive closed point sets $H$ and $L$. These point sets both intersect $M - K$. Since the continuum $K$ is compact it contains a connected point set $T$ lying in $S - M$ and such that $T$ intersects both $H - M - K$ and $L - M - K$. Let $A$ and $B$ denote points belonging, respectively, to the subsets $T - H - M - K$ and $T - L - M - K$ of $\beta$. There exists a compact continuum $M'$ lying in $M$ and containing $A$ and $B$. Suppose $P$ is a point of $M' - M' - \beta$. Let $D_p$ denote the component of $M - \beta$ that contains $P$ and let $E_p$ denote the component of $S - D_p$ that contains $T$. By hypothesis, the boundary of $E_p$ is a subset of a component $\beta_p$ of $\beta$. For each point $P$ of $M' - M' - \beta$, $\beta_p$ shields $A + B$ from $P$ in $M$. Hence, by Theorem 6 of C. B., there is a continuum containing $A$ and $B$ and lying in $\beta$. This involves a contradiction.

**Theorem 13.** Theorem 12 remains true if the requirement that $K$ be compact is replaced by the requirement that $M - K$ be compact and that space satisfy Axiom 2.

Theorem 13 may be proved by an argument identical with that given to prove Theorem 12 except for the substitution of “By Theorem 6 of D. C. B., there exists” for “Since the continuum $K$ is compact it contains” in the third sentence of that argument.

Theorem 12 does not remain true on the omission of the stipulation that $M$ is compactly connected even though it is stipulated that $K$ contains $\beta$. Consider the following example.

**Example 3.** In a Cartesian plane $E$, let $O$ denote the origin and, for each positive integer $n$, let $A_n$, $B_n$, $C_n$, $D_n$ and $E_n$ denote the points $(0, 1/n), (1/n, 1/n), (1/n, -1/n), (-1/n, -1/n)$ and $(-1/n, 1)$ and let $F_n$ denote a point lying midway between $B_n$ and $A_{n+1}$. If $X$ and $Y$ are two points let $XY$ denote the straight line interval whose extremities are $X$ and $Y$. Let $t_n$ denote the point set $A_nB_n + B_nC_n + C_nD_n + D_nE_n$. Let $M$ and $K$ denote the point sets $O + t_1 + t_2 + t_3 + \ldots$ and $O + A_1F_1 + F_1A_2 + A_2F_2 + F_2A_3 + \ldots$, respectively. Let $S$ denote the point set $M + K$ and let $\Sigma$ denote the subspace of $E$ whose points are the points of $S$. Here $\beta$ is the point set $O + A_1 + A_2 + \ldots$ and $K$ is a compact con-
tinuum containing $\beta$. Furthermore if $D$ is a component of $M - \beta$ then, for some $n$, $D$ is $t_n - A_n$ and if $E$ is a component of $S - D_n$ then $E$ is $S - t_n$ and its boundary is the point $A_2$ which is a connected subset of $\beta^D$. But $M \cdot K = \beta$ and every component of $M \cdot K + \beta$ is a point of the infinite point set $\beta$. Hence $M \cdot K$ is not a subset of any component of $M \cdot K + \beta$.

Neither does Theorem 12 hold true if the stipulation that $\beta$ is compact is replaced by the stipulation that Axiom 2 holds true. Consider the following example.

**Example 4.** In a Cartesian space $E$ of three dimensions, for each positive integer $n$, let $T_n$ denote the portion of the graph of $y = -n + 1/ (x - x^2)$ that lies in the $XY$ plane between the planes $x = 0$ and $x = 1$. For each $n$, let $S_n$ denote the set of all points $(x, y, z)$ of this graph such that $0 < x < 1$ and $0 < z < 1/n$. Let $A$, $B$ and $C$ denote the points $(0, 0, 2), (1, 0, 2)$ and $(1, 0, 0)$, respectively. Let $K$ denote the arc obtained by adding together the straight line intervals $OA$, $AB$ and $BC$. Let $M$ denote the $XY$ plane and let $S$ denote $K + M + S_1 + S_2 + S_3 + \ldots$. Let $\Sigma$ denote the subspace of $E$ whose points are the points of $S$. Here $K$ is compact and $M \cdot K = O + C$. The boundary of $M$ is the point set obtained by adding together the open curves $T_1$, $T_2$, $T_3$, $\ldots$, the $Y$-axis and a line through $C$ parallel to the $Y$-axis. The point set $M \cdot K + \beta$ is identical with $\beta$ and no component of $\beta$ contains both $O$ and $C$. It is to be noted that $\beta$ is not the sum of two mutually separated point sets containing $O$ and $C$, respectively.

However, the following theorem holds true.

**Theorem 14.** If, in a space satisfying Axioms 0, 1, and 2, the continuum $K$ contains $\beta$, the boundary of the continuum $M$, and, for every component $D$ of $M - \beta$ and component $E$ of $S - D$, the boundary of $E$ is connected then $M \cdot K$ is connected.

**Proof.** Suppose $M \cdot K$ is the sum of two mutually exclusive closed point sets $H$ and $L$. By Theorem 11, $H \cdot \beta$ and $L \cdot \beta$ exist. Furthermore, by hypothesis, $\beta$ is a subset of their sum. Hence, by Theorem 5 of C. B., there exists a component $D$ of $M - \beta$ such that $D$ intersects both $H$ and $L$. The common part of the continua $D$ and $K$ is the sum of the two mutually exclusive closed point sets $D \cdot K \cdot H$ and $D \cdot K \cdot L$. Therefore, by Theorem 6 of D. B. C., there exists an arc $AB$ from the point $A$ of $D \cdot K \cdot H$ to the point $B$ of $D \cdot K \cdot L$ and having no point in common with $D$ except its end-points $A$ and $B$. Let $E$ denote the component of $S - D$ which contains $AB - (A + B)$. The boundary of $E$ is a connected subset of $\beta$ containing the points $A$ and $B$. This involves a contradiction.

**Theorem 15.** If, in a space satisfying Axioms 0 and 1, $\beta$ is the boundary of the continuum $M$, $D$ is a component of $M - \beta$, $K$ is a compact continuum intersecting $D$ but not lying wholly in it and $Q$ is a component of $K-(S - D)$ then every component of $Q \cdot D$ contains a point of $\beta$. 


Proof. Suppose the component $H$ of $\overline{Q-D}$ contains no point of $\beta$. Let $O$ denote some point of $Q - H$. Since $\overline{Q}$ is a compact continuum there exists a domain $W$ containing $H$, but no point of $O + \beta$, and such that its boundary $\gamma$ contains no point of $\beta + \overline{Q-D}$. The point set $\overline{Q-W}$ contains a connected point set $T$ such that $\overline{T}$ intersects $\gamma$ and contains a point $P$ of $\overline{D}$. Since $P$ does not belong to $\beta$, it belongs to $D$. But $T + P$ is a connected point set containing no point of $\beta$. Hence $T + P$ is a subset of $D$. Since $T$ is a subset of $Q$ this involves a contradiction.

Theorem 16. If, in a space satisfying Axioms 0 and 1, $\beta$ is the boundary of the compact continuum $M$ and $D$ is a component of $M - \beta$ and, for every component $E$ of $S - D$, the common part of $\beta$ and the boundary of $E$ is connected and $K$ is a compact continuum intersecting $\overline{D}$ then $K \cdot D$ is a subset of a component of $K \cdot D + \beta$.

Proof. For every component $E$ of $S - D$ that intersects $K$, let $T_E$ denote the common part of $\beta$ and the boundary of $E$. With the help of Theorem 15 it may be seen that if $Q$ is a component of $K \cdot E$ every component of $Q \cdot D$ intersects $\beta$ and therefore $T_E$. Hence, by Theorem 5, if $T$ is the sum of the continua $T_E$ for all $E$'s, $K \cdot D + T$ is connected. But $T$ is a subset of $\beta$.

Theorem 17. If, in a space satisfying Axioms 0 and 1, $\beta$ is the boundary of the compact continuum $M$ and, for every component $D$ of $M - \beta$ and component $E$ of $S - D$, the common part of $\beta$ and the boundary of $E$ is connected and $K$ is a compact continuum containing $\beta$ then $M \cdot K$ is a continuum.

Proof. If $D$ is a component of $M - \beta$, the continuum $K$ intersects $\overline{D}$ and therefore, by hypothesis and Theorem 16, $K \cdot D$ is a subset of a component $T_D$ of $K \cdot D + \beta$. Let $T$ denote the sum of all the point sets $T_D$ for all $D$'s. By Theorem 1 of C. B., $\beta + T$ is connected. But this point set is identical with $M \cdot K$.

Theorem 17 remains true if the stipulation that the common part of $\beta$ and the boundary of $E$ is connected is replaced by the stipulation that it is a subset of a component of $M \cdot K$. But it does not remain true if "the common part of $\beta$ and the boundary of $E$" is replaced by "the common part of $\overline{D}$ and the boundary of $E$" or by "the boundary of $E." Consider the following example.

Example 5. In a Euclidean plane $E$ let $\alpha$ denote a definite square and let $\gamma$ denote a definite square enclosed by $\alpha$. Let $O$ denote the midpoint of one side of $\gamma$ and let $Q$ denote a totally disconnected perfect point set lying on the opposite side of $\gamma$. Let $M$ denote the point set obtained by adding together all straight line intervals with one end-point at $O$ and the other one at a point of $Q$. Let $K$ denote the sum of $\alpha$ and $\gamma$ and the set of all points that lie between them. Let $S$ denote $M + K$ and let $E$ denote the subspace of $E$ whose points are the points of $S$. Here $S$ is compact, $\beta$ is $O + Q$, $K$ contains $\beta$ and, for every component $D$ of $M - \beta$,
CONCERNING A CONTINUUM AND ITS BOUNDARY

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Communicated October 23, 1942

In this paper some theorems will be established concerning certain relationships between the boundary of a continuum and the components of that continuum minus its boundary. Numbered axioms and chapters herein referred to are axioms and chapters of the author's book "Foundations of Point Set Theory."\(^1\)

**Theorem 1.** If, in a space satisfying Axioms 0 and 1, \(\beta\) is the boundary of the compact continuum \(M\) and, for every component \(D\) of \(M - \beta\), \(T_D\) is a connected point set lying in \(M\) and containing the common part of \(\beta\) and the boundary of \(D\), and \(T\) is the sum of all the point sets \(T_D\) for all such \(D's\), then \(\beta + T\) is connected.

**Proof.** Suppose \(\beta + T\) is the sum of two mutually separated point sets \(H\) and \(K\). The point sets \(H \cdot \beta\) and \(K \cdot \beta\) exist and are mutually exclusive and closed. There exists a subcontinuum \(N\) of \(M\) which is irreducible from \(H \cdot \beta\) to \(K \cdot \beta\). The connected point set \(N - (H \cdot \beta + K \cdot \beta)\) is a subset of some component \(L\) of \(M - \beta\). Each of the point sets \(H \cdot \beta\) and \(K \cdot \beta\) con-