must look for the effective agent in this change. In this respect the effect of temperature is the opposite to that found in Trillium and Triton. That in itself is not surprising, for in them the metaphase charge is concerned while in Drosophila it is the resting stage reproduction. And between resting stage and metaphase there is a reversal of charge in Trillium and Triton. Thus the two parts of the relationship: temperature—nucleic acid charge—division cycle, are seen pieced together, although how they are pieced together we do not yet know.

Summary.—The "sex-ratio" gene or complex in the X-chromosome of many Drosophila species causes the X-chromosome to divide twice at meiosis in the males while the Y is thrown out. This abnormality seems to depend on an excessive nucleic acid charge and should therefore (on other evidence) be affected by temperature. This expectation was confirmed. The proportion of X-sperm was reduced from 99 to 94 per cent by raising the temperature from 16°C to 25°C.

* Experimental data by Th. Dobzhansky, the general interpretation by C. D. Darlington.
4 Dobzhansky, Th., *Scientia Genetica*, 1, 67–75 (1939).

A GENERAL THEOREM ON THE INITIAL CURVATURE OF DYNAMICAL TRAJECTORIES*

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The theorem we intend to establish is an extension to acceleration fields of higher order of Kasner's dynamical theorem which states that: *If a particle starts from rest in any positional field of force, the initial curvature of the trajectory is one-third of the curvature of the line of force through the initial position.*

We formulate this result analytically. If \( \phi(x, y) \) and \( \Psi(x, y) \) represent the rectangular components of a planar field of force acting at any point...
If the curvature of the lines of force is everywhere zero, then the lines of force are straight lines, and a particle starting from rest at any point must necessarily move along the line of force. Thus, the trajectory and the line of force coincide and the theorem is trivially true. Excluding this degenerate possibility, we choose a point \((x_0, y_0)\) at which the curvature of the line of force, which will be represented by \(\Omega\), is not zero. By direct calculation, we find that the curvature of the line of force, which line of force satisfies the differential equation \(\frac{dy}{dx} = \frac{\Psi(x, y)}{\varphi(x, y)}\), is

\[
\Omega = \frac{\varphi(\varphi_x \varphi + \varphi_y \Psi) - \Psi(\varphi_x \varphi + \varphi_y \Psi)}{(\varphi^2 + \Psi^2)^{3/2}},
\]

where the subscripts \(x\) and \(y\) denote partial differentiation and the functions \(\varphi, \Psi\) and their partial derivatives are evaluated at \((x_0, y_0)\). If \(\gamma_2(x_0, y_0)\) is the curvature of the trajectory of the particle which starts from rest at \((x_0, y_0)\), then Kasner’s theorem states

\[
\gamma_2(x_0, y_0) = \frac{1}{3\Omega}.
\]

The result we wish to prove is the following generalization.

**Theorem:** If a particle starts from “maximum rest” in an acceleration field of order \(n\), the initial curvature of the trajectory is \(\frac{n!(n - 1)!}{(2n - 1)!}\) times the curvature of the line of force through the initial position.

Again, if \(\varphi(x, y)\) and \(\Psi(x, y)\) are the rectangular components of a planar acceleration field of order \(n\) acting at any point \((x, y)\) and \(t\) is the time, then the equations of motion for a particle of unit mass are

\[
\frac{d^n x}{dt^n} = \varphi(x, y), \quad \frac{d^n y}{dt^n} = \Psi(x, y).
\]

By the phrase “maximum rest,” we mean that for \(t = 0\), the initial conditions are

\[
\frac{d^j x}{dt^j} = \frac{d^j y}{dt^j} = 0, \quad 0 < j < n;
\]

that is, not merely is the initial velocity zero, but also all the initial accelerations of order up to and including \(n - 2\) are zero.
As before, if the curvature of the lines of force is everywhere zero, then the lines of force are straight lines, and a particle starting from maximum rest at any point must move along the line of force through that point. Thus the trajectory and the line of force coincide and the theorem will be trivially true. Excluding this possibility, we choose a point \((x_0, y_0)\) at which the curvature of the line of force is not zero. The curvature of the trajectory at \((x_0, y_0)\) may be calculated from the formula

$$
\gamma_n = \frac{\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right]^{1/2}} \tag{3}
$$

However, as \(\frac{dx}{dt} = \frac{dy}{dt} = 0\) for \(t = 0\), \(\gamma_n\) assumes an indeterminate form and we must have recourse to the theory of limits in order to obtain the desired value. To achieve this, we expand the solution of (2) as power series in the time.

Let the parametric equations of the trajectory be

\[
\begin{align*}
x &= a_0 + a_1 t + \ldots + \frac{a_r}{r!} t^r + \ldots \\
y &= b_0 + b_1 t + \ldots + \frac{b_r}{r!} t^r + \ldots
\end{align*}
\]

where the coefficients \(a_r, b_r\) are to be evaluated. From the initial conditions it follows immediately that

\[
\begin{align*}
a_0 &= x_0 & a_1 = a_2 = \ldots = a_{n-1} &= 0, & a_n &= \varphi(x_0, y_0) \\
b_0 &= y_0 & b_1 = b_2 = \ldots = b_{n-1} &= 0, & b_n &= \Psi(x_0, y_0).
\end{align*}
\]

Differentiate equations (2) with respect to the time. This yields

\[
\frac{d^{n+1}x}{dt^{n+1}} = \varphi_x \frac{dx}{dt} + \varphi_y \frac{dy}{dt}, \quad \frac{d^{n+1}y}{dt^{n+1}} = \Psi_x \frac{dx}{dt} + \Psi_y \frac{dy}{dt}.
\]

Therefore, for \(t = 0\),

\[
\frac{d^{n+1}x}{dt^{n+1}} = 0, \quad \frac{d^{n+1}y}{dt^{n+1}} = 0
\]

since \(\frac{dx}{dt} = 0\) and \(\frac{dy}{dt} = 0\). By repeated differentiations we find as initial values for the higher derivatives.
\[
\frac{d^n + j \cdot x}{dt^n + j} = 0, \quad \frac{d^n + j \cdot y}{dt^n + j} = 0
\]

for all values of \( j \) less than \( n \). However, for \( j = n \), the appropriate initial values are

\[
\frac{d^{2n}x}{dt^{2n}} = \varphi_{x\varphi} + \varphi_{y\Psi}, \quad \frac{d^{2n}y}{dt^{2n}} = \Psi_{x\varphi} + \Psi_{y\Psi}
\]

with \( \varphi, \Psi \) and their partial derivatives evaluated at \((x_0, y_0)\). Therefore,

\[
a_{n+1} = \ldots = a_{2n-1} = 0, \quad a_{2n} = \varphi_{x\varphi} + \varphi_{y\Psi}
\]

\[
b_{n+1} = \ldots = b_{2n-1} = 0, \quad b_{2n} = \Psi_{x\varphi} + \Psi_{y\Psi}.
\]

and the expansions for the equations of the trajectory are

\[
x = x_0 + \frac{\varphi}{n!} t^n + \frac{\varphi_{x\varphi} + \varphi_{y\Psi}}{(2n)!} t^{2n} + \text{higher powers}
\]

\[
y = y_0 + \frac{\Psi}{n!} t^n + \frac{\Psi_{x\varphi} + \Psi_{y\Psi}}{(2n)!} t^{2n} + \text{higher powers}.
\]

Substituting in formula (3) for the curvature, we obtain

\[
y_n = \frac{1}{(n-2)!(2n-1)!} \left[ \varphi^2 + \Psi^2 \right]^{\frac{1}{\gamma_n}} \left[ \frac{1}{(n-1)!} \right]^{3 + \text{higher powers}}.
\]

Taking the limit as \( t \to 0 \), and comparing with formula (1), we have the desired conclusion, namely

\[
y_n(x_0, y_0) = \frac{n!(n-1)!}{(2n-1)!} \Omega.
\]

This completes the proof.

For the value \( n = 2 \), we obtain as a corollary Kasner's theorem,

\[
y_2(x_0, y_0) = \frac{1}{3} \Omega.
\]

For \( n = 3 \), we find \( y_3 = 1/10 \Omega \); for \( n = 4 \), \( y_4 = 1/35 \Omega \).

We notice that the ratio of any two successive values of the initial curvature takes the simple form

\[
\frac{\gamma_{n+1}}{\gamma_n} = \frac{n + 1}{2(2n + 1)}.
\]
Our general theorem (as in the special\textsuperscript{1} case $n = 2$) is found to be valid in ordinary space of any number of dimensions and also in riemannian geometry.

\textsuperscript{*} Presented to the American Mathematical Society, February, 1942.


\section*{GENERALIZED TRANSFORMATION THEORY OF ISOThERMAL AND DUAL FAMILIES\textsuperscript{*}}

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The Isothermal Character Theorem.—A noteworthy theorem in the theory of functions of a complex variable is that the conformal transformations are the only point correspondences which carry every isothermal family of curves into an isothermal family. Kasner's generalization\textsuperscript{1} of this result to the lineal-element transformations of the plane is

\textbf{Theorem 1.} The group of lineal-element transformations of the real plane which convert every isothermal family into an isothermal family is in cartesian coordinates $(x, y, \theta)$

\begin{equation}
X = \phi(x, y), \quad Y = \psi(x, y), \quad \Theta = a\theta + h(x, y),
\end{equation}

where $\phi$ and $\psi$ satisfy the Cauchy-Riemann equations (direct or reverse)

\begin{equation}
\phi_x = \pm \psi_y, \quad \phi_y = \mp \psi_x,
\end{equation}

and $h$ is any harmonic function of $(x, y)$, and $a$ is a non-zero constant.

A corollary of this is that the only contact lineal-element transformations preserving the isothermal character are the conformal.

This theorem contains most well-known devices for obtaining isothermal families from a given isothermal family. In particular, there appear as special cases the facts that the isogonals and the multiplicatives of an isothermal family each form an isothermal family. However Theorem 1 does not contain the less familiar result that the isoclines of an isothermal family are an isothermal family. Our new generalization to field-element transformations will contain this fact as a special case.