RELATIVE NUMBER OF NON-ININVARIANT OPERATORS IN A GROUP

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Communicated December 30, 1943

Since the number of the invariant operators of a given group \( G \), of order \( g \) is equal to the order of the central \( c \) of \( G \), the number of the non-invariant operators of \( G \) is \( g - c \). In every non-abelian group the central quotient group is known to be non-cyclic and hence its order is a composite divisor of the order of \( G \). In particular, when the order of \( G \) is the product of two distinct prime numbers and \( G \) is non-abelian the number of the non-invariant operators of \( G \) is \( g - 1 \), and when the order of the non-abelian group \( G \) is \( p^3 \), \( p \) being a prime number, then the number of the non-invariant operators of \( G \) is \( p^3 - p = p(p^2 - 1) \). A non-abelian group of order \( g \) can therefore not contain less non-invariant operators than \( g \) minus \( g \) divided by the two smallest positive integers which divide the order of \( g \).

From the preceding paragraph it results that a non-abelian group of order \( g \) cannot have less than \( 3g/4 \) non-invariant operators and that it must involve a larger number of such operators when \( g \) is not divisible by 4. In fact, it must involve a larger number of such operators unless \( g \) is divisible by 8 as we proceed to prove. Suppose that \( G \) contains exactly \( 3g/4 \) non-invariant operators and hence its central quotient group is the four group. Its central will then give rise to three co-sets corresponding to the three operators of order 2 in the four group. If \( s_1 \) is an operator from one of these co-sets and \( s_2 \) is an operator from another of these three co-sets then \( s_1 \) and \( s_2 \) are necessarily non-commutative sines; otherwise they would appear in the central of \( G \).

A commutator which arises from \( s_1 \) and \( s_2 \) must be in the central of \( G \) and it must be of order 2 since the squares of all of the operators of \( G \) must appear in the central of \( G \). It therefore results that the central of \( G \) is of even order and hence the order of \( G \) is divisible by 8. It is obvious that each of the two non-abelian groups of order 8 satisfies the condition that the number of the non-invariant operators contained therein is \( 3g/4 \), where
$g$ is the order of the group. Moreover the direct product of one of these
groups of order 8 and an arbitrary abelian group evidently satisfies the
same condition. Hence there results the following theorem: *A necessary
and sufficient condition that there is at least one group of order $g$ in which
the number of the non-invariant operators is exactly $3g/4$ is that $g$ is divisible
by 8 and there is no non-abelian group of order $g$ which contains a smaller
number of non-invariant operators.* This theorem can clearly be extended.

When the number of the non-invariant operators of $G$ is exactly $3g/4$
all the operators of odd order contained in $G$ must appear in its central.
Hence such a $G$ is the direct product of its Sylow subgroups and all of its
Sylow subgroups of odd order are abelian while its Sylow subgroup whose
order is a power of 2 has a commutator subgroup of order 2 as was noted
above. The determination of the groups which involve exactly $3g/4$
non-invariant operators is therefore reduced to the determination of such
groups having an order which is a power of 2. In particular, if a group
whose order is not divisible by 16 has the property that it contains exactly
$3g/4$ non-invariant operators it is the direct product of one of the two non-
abelian groups of order 8 and an arbitrary abelian group of odd order, and
all such direct products have the property that the number of the non-
invariant operators contained in each of them is $3g/4$, where $g$ is the order
of the group.

If $G$ contains exactly $3g/4$ non-invariant operators then it contains three
abelian subgroups of order $2^m$, $m > 1$, which have $2^{m-1}$ operators in com-
mon and these common operators are contained in the central of $G$ while
the Sylow subgroup whose order is a power of 2 is of order $2^{m+1}$. The com-
mutator subgroup of order 2 of $G$ appears in the central of $G$ but it is not
necessarily generated by an operator of higher order in $G$ as results from a
non-abelian group of order 16 in which two of the three abelian subgroups
of order 8 are of type 2, 1 and the central is the four group. Two of the
three operators of order 2 in this four group are squares of operators of
order 4 contained therein while the third is a commutator of the group.
When the number of the non-invariant operators of a group is $3g/4$ the
smallest number of these operators is 6 since it must be an even number
and the two non-abelian groups of order 8 are the only groups which have
separately exactly 6 non-invariant operators.

Every group of order $g$ which has the property that exactly $3g/4$ of its
operators are non-invariant contains a multiple of 6 non-invariant opera-
tors and it is possible to construct such a group in which the total number
of non-invariant operators is an arbitrary multiple of 6. This results
directly from the fact that in the direct product of a group which involves
exactly 6 such operators and an abelian group of arbitrary order the
number of the non-invariant operators is equal to 6 times this arbitrary
order. In particular, the groups which contain exactly 12 non-invariant operators are of order 16 and have a central of order 4. There is one and only one such group which contains the cyclic subgroup of order 8. The remaining operators of this group transform each operator of the cyclic group into its fifth power. There is obviously another such group in which the central is the cyclic group of order 4 and there are four such groups in which the central is the non-cyclic group of order.

There is only a finite number of different non-abelian groups which separately have the property that each of them contains the same number of non-invariant operators. This number of groups may be zero. The fact that there is only a finite number of such different groups is a direct consequence of the theorem that there is an upper limit for the order of such groups since this order cannot be as large as twice the number of the non-invariant operators contained therein. On the contrary, the number of the different groups which contain separately the same given number of non-invariant subgroups is not necessarily limited by this number of the non-invariant subgroups contained in these separate groups. In fact, it is known that there is an infinite system of groups such that each of them contains two and only two non-invariant subgroups. (Cf. these Proceedings, 29, 105 (1943).) There is, however, no group which contains exactly two non-invariant operators as will be more fully explained.

If the number of the non-invariant operators of $G$ exceeds $3g/4$ it is at least $5g/6$ because the order of the central quotient group is then at least 6 and if this order is 6 the central quotient group is the non-cyclic group of this order. The direct product of this group and an arbitrary abelian group evidently furnishes an infinite system of groups such that the number of the non-invariant operators contained in each of these groups is $5g/6$, $g$ being the order of the group. The number of the non-invariant operators in each of these groups is a multiple of 5 and there is at least one group in which this number is an arbitrary multiple of 5. The symmetric group of order 6 is the only one of these groups which contains exactly 5 non-invariant operators and this group is characterized by the fact that it contains exactly 5 non-invariant operators.

A necessary and sufficient condition that there exists a group in which the number of non-invariant operators is exactly a given prime number $p$ is that there exists a group of order $p + 1$ which contains no invariant operator besides the identity. This theorem results from the fact that this prime number is equal to the order of the central quotient group diminished by unity and that this central quotient group cannot involve any invariant operator besides the identity. It therefore results that there exists no group in which the number of the non-invariant operators is exactly 3, 7 or 31, but that there exists at least one non-abelian group in which the number of the non-invariant operators is any one of the prime numbers in the
following set: 5, 11, 13, 17, 19, 23, 29. It would be a very simple matter to extend these sets.

It is now easy to prove the following theorem: Every group in which the total number of non-invariant operators is a prime number contains no invariant operator besides the identity. In fact, if the total number of the non-invariant operators of the group is a prime number the order of the central quotient group diminished by unity must be equal to this prime number. The central quotient group must be simply isomorphic with the group since the number of the non-invariant operators of a group is the product of the order of the central, and the order of the central quotient group diminished by unity. The converse of this theorem is obviously not necessarily true since it is easy to find groups which contain no invariant operator besides the identity but in which the number of the non-invariant operators is not a prime number. For instance, the dihedral group of order 10 has this property.

There is one and only one group in which the total number of the non-invariant operators is one of the three prime numbers 5, 11, 13 but there are two groups in which the number of the non-invariant operators is exactly 17. One of these is the dihedral group of order 18 and the other is the generalized dihedral group of this order. When the number of the non-invariant operators contained in $G$ is exactly 23 the symmetric group of order 24 is the only one of the 15 groups of this order which contains no invariant operator besides the identity and hence this group is characterized by the fact that it contains exactly 23 non-invariant operators. This characterization of the symmetric group of degree 4 may therefore be added to the numerous known definitions of this well-known group.

NOTE ON THE FUNCTION $F(a, b; c - n; z)$

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Communicated January 8, 1944

The generating function

$$(1 - t)^{b - 1}(1 - z + zt)^{-a} = \sum_{n=0}^{\infty} (\frac{n}{n!})(1 - b, n)F(a, b; b - n; z) |t| < 1, |zt| < |1 - z|$$

may be used to find an estimate of $F(a, b; b - n; z)$ for large positive values of $n$. When the point $t = 1 - 1/z$ lies outside the circle $t = 1$ the singularity $t = 1$ may be used to find an estimate by the method of Darboux$^1$ and the result is