lations when the number of new cases is less or only slightly greater than the number of new susceptibles during each incubation period.


A TOPOLOGY FOR THE SET OF PRIMITIVE IDEALS IN AN ARBITRARY RING

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1. In a recent paper we have called a ring $\mathfrak{A}$ primitive if $\mathfrak{A}$ contains a maximal right ideal $\mathfrak{I}$ whose quotient $\mathfrak{I}:\mathfrak{A} = 0$. In general if $\mathfrak{I}$ is any right ideal $\mathfrak{I}:\mathfrak{A}$ is the totality of elements $b$ such that $xb \in \mathfrak{I}$ for all $x$ in $\mathfrak{A}$. $\mathfrak{I}:\mathfrak{A}$ is a two-sided deal and if $\mathfrak{A}$ has an identity, $\mathfrak{I}:\mathfrak{A}$ is the largest two-sided ideal of $\mathfrak{A}$ contained in $\mathfrak{I}$. The primitive rings appear to play the same rôle in the general structure theory of rings that is played by simple rings in the classical theory of rings that satisfy the descending chain condition for one-sided ideals. Corresponding to the Wedderburn-Artin structure theorem on simple rings satisfying the descending chain condition we have the theorem that if $\mathfrak{A}$ is a primitive ring $\neq 0$, $\mathfrak{A}$ is isomorphic to a dense ring of linear transformations in a suitable vector space over a division ring.

We shall call a two-sided ideal $\mathfrak{B}$ in $\mathfrak{A}$ a primitive ideal if $\mathfrak{B} \neq \mathfrak{A}$ and $\mathfrak{A} - \mathfrak{B}$ is a primitive ring. It is known that if $\mathfrak{A}$ is not a radical ring then $\mathfrak{A}$ contains primitive ideals. Moreover, in this case the intersection $\Pi \mathfrak{B}$ of all the primitive ideals in $\mathfrak{A}$ coincides with the radical $\mathfrak{R}$ of $\mathfrak{A}$. In particular if $\mathfrak{A}$ is semi-simple, $\Pi \mathfrak{B} = 0$. If $\mathfrak{A}$ is a ring with an identity, $\mathfrak{A}$ is not a radical ring. Hence if $\mathfrak{C}$ is any two-sided ideal $\neq \mathfrak{A}$ in $\mathfrak{A}$, $\mathfrak{A} - \mathfrak{C}$ contains a primitive ideal $\mathfrak{B} - \mathfrak{C}$. Since $\mathfrak{A} - \mathfrak{B} \cong (\mathfrak{A} - \mathfrak{C}) - (\mathfrak{B} - \mathfrak{C})$, $\mathfrak{B}$ is primitive in $\mathfrak{A}$. Thus if $\mathfrak{A}$ is a ring with an identity any two-sided ideal $\mathfrak{C} \neq \mathfrak{A}$ can be imbedded in a primitive ideal.

Any commutative primitive ring is a field. Consequently any primitive ideal in a commutative ring is maximal. On the other hand, in a non-commutative ring there may exist primitive ideals that are not maximal. For example, the ring $\mathfrak{L}$ of all linear transformations in an infinite dimensional vector space $\mathfrak{H}$ over a division ring is primitive. Hence $(0)$ is a primitive ideal. However, $\mathfrak{L}$ contains as a proper two-sided ideal the set $\mathfrak{J}$ of finite valued linear transformations in $\mathfrak{H}$.
A simple ring is either primitive or a radical ring. In particular any simple ring with an identity is primitive. These remarks imply that a maximal two-sided ideal \( \mathfrak{B} \) such that \( \mathfrak{A} - \mathfrak{B} \) is not a radical ring is primitive. If \( \mathfrak{A} \) is a ring with an identity, any maximal two-sided ideal in \( \mathfrak{A} \) is primitive.

In this note we shall define a topology for the set of primitive ideals of any ring. The space determined in this way appears to be an important invariant of the ring. We hope to discuss its rôle in the general structure theory in greater detail at a later date.

2. Let \( S \) be the set of primitive ideals in the ring \( \mathfrak{A} \). \( S \) is vacuous if and only if \( \mathfrak{A} \) is a radical ring. If \( A \) is a non-vacuous subset of \( S \) we let \( \mathcal{D}_A \) denote the intersection of all the primitive ideals \( \mathfrak{B} \in A \). We now define the closure \( \overline{A} \) of \( A \) to be the totality of primitive ideals \( \mathfrak{C} \) such that \( \mathfrak{C} \geq \mathcal{D}_A \). It is clear that

1. \( \overline{A} \supseteq A \).
2. \( \overline{A} = A \).

We wish to prove next that

3. \( \overline{A} \lor B = \overline{A} \lor \overline{B} \).

Proof. Let \( \mathfrak{B} \in \overline{A} \lor B \), say \( \mathfrak{B} \in \overline{A} \lor B \). Then \( \mathfrak{B} \supseteq \mathcal{D}_A \lor \mathcal{D}_B = \mathcal{D}_C \) where \( C = A \lor B \). Thus \( \mathfrak{B} \in A \lor B \). Suppose next that \( \mathfrak{B} \in A \lor B \). Then \( \mathfrak{B} \supseteq \mathcal{D}_A \) and \( \mathfrak{B} \supseteq \mathcal{D}_B \). We consider now the primitive ring \( \mathfrak{A} - \mathfrak{B} \). The two-sided ideals \( (\mathcal{D}_A + \mathfrak{B}) - \mathfrak{B} \) and \( (\mathcal{D}_B + \mathfrak{B}) - \mathfrak{B} \) are \( \neq 0 \) in \( \mathfrak{A} - \mathfrak{B} \). Hence the product \( [(\mathcal{D}_A + \mathfrak{B}) - \mathfrak{B}] [(\mathcal{D}_B + \mathfrak{B}) - \mathfrak{B}] \neq 0 \). It follows that \( \mathcal{D}_A \mathcal{D}_B \subseteq \mathfrak{B} \). Hence also \( \mathcal{D}_A \lor \mathcal{D}_B \subseteq \mathfrak{B} \) and so \( \mathfrak{B} \in A \lor B \).

For the vacuous set \( \omega \) we define

4. \( \overline{\omega} = \omega \).

The properties 1–4 show that \( S \) is a topological space relative to the closure operation \( A \rightarrow \overline{A} \). In the special case of a Boolean ring this topology is due to Stone. It has also been introduced by Gelfand and Silov in commutative normed rings. We shall call the topological space \( S \) the structure space of the ring \( \mathfrak{A} \).

If \( \mathfrak{B} \) is a primitive ideal in \( \mathfrak{A} \), \( \mathfrak{B} \) is a point in \( S \). The closure of the set \( \{ \mathfrak{B} \} \) is the totality of primitive ideals \( \mathfrak{C} \) such that \( \mathfrak{C} \supseteq \mathfrak{B} \). Hence if \( \{ \mathfrak{B}_1 \} = \{ \mathfrak{B}_2 \} \) then \( \mathfrak{B}_1 = \mathfrak{B}_2 \). This shows that \( S \) is a \( T_0 \)-space. In general \( S \) is not a \( T_1 \)-space. For we have seen that there exist primitive rings \( \mathfrak{A} \) that are not simple. If \( \mathfrak{A} \) is a ring with an identity of this type and \( \mathfrak{B} \) is a proper two-sided ideal in \( \mathfrak{A} \), \( \mathfrak{B} \) can be imbedded in a primitive ideal \( \mathfrak{C} \). Then \( \{ (\mathfrak{O}) \} \) contains \( \mathfrak{C} \neq 0 \). The subspace \( M \) of \( S \) of primitive ideals that are maximal is clearly a \( T_1 \)-space. If \( \mathfrak{A} \) is commutative \( S = M \) is a \( T_1 \)-space. However, even in this case \( S \) need not be a \( T_2 \)- (or Hausdorff) space. An example of a normed ring of this type has been given by Gelfand and Silov. A simpler one is the following:
Example. Let \( \mathfrak{A} = J \) the ring of integers. The primitive ideals are the prime ideals \( (p) \). Since the intersection of an infinite number of prime ideals is the 0-ideal, \( \mathfrak{A} = S \) for any infinite set \( A \). If \( A \) is finite, \( \mathfrak{A} = A \). Hence the open sets \( \neq \omega, \neq S \) are the complements of finite sets. Any two open sets \( \neq \omega \) have a non-vacuous intersection and so the Hausdorff separation property does not hold.

3. Let \( \mathfrak{A}_1 \) be an arbitrary two-sided ideal in \( \mathfrak{A} \) and let \( S_1 \) denote the closed set in \( \mathfrak{A} \) consisting of the primitive ideals \( \mathfrak{B} \) of \( \mathfrak{A} \) that contain \( \mathfrak{A}_1 \). If \( \mathfrak{B} \in S_1 \) then \( \mathfrak{B} - \mathfrak{A}_1 \) is a primitive ideal in \( \mathfrak{A} - \mathfrak{A}_1 \) and any primitive ideal in \( \mathfrak{A} - \mathfrak{A}_1 \) is obtained in this way. The correspondence \( \mathfrak{B} \to \mathfrak{B} - \mathfrak{A}_1 \) is \((1 - 1)\) between the subspace \( S_1 \) of \( \mathfrak{A} \) and the structure space \( T \) of the ring \( \mathfrak{A} - \mathfrak{A}_1 \). Since this correspondence preserves intersection it is a homeomorphism between \( S_1 \) and \( T \).

Suppose in particular that \( \mathfrak{A}_1 = \mathfrak{R} \) the radical of \( \mathfrak{A} \). Then every primitive ideal of \( \mathfrak{A} \) contains \( \mathfrak{R} \). Hence \( S_1 = S \), and we see that the structure space of \( \mathfrak{A} \) is homeomorphic to the structure space of the semi-simple ring \( \mathfrak{A} - \mathfrak{R} \).

We return to the general case in which \( \mathfrak{A}_1 \) is arbitrary and we now consider the open set \( S'_1 \) of primitive ideals that do not contain \( \mathfrak{A}_1 \). Let \( \mathfrak{C} \in S'_1 \). Then \( \mathfrak{C} \wedge \mathfrak{A}_1 \) is a two-sided ideal \( \neq \mathfrak{A}_1 \) in \( \mathfrak{A}_1 \) and \( \mathfrak{A}_1 - (\mathfrak{C} \wedge \mathfrak{A}_1) \cong (\mathfrak{C} + \mathfrak{A}_1) - \mathfrak{C} \). The latter ring is a two-sided ideal in the primitive ring \( \mathfrak{A} - \mathfrak{C} \). Hence it is primitive. Thus \( \mathfrak{C} \wedge \mathfrak{A}_1 \) in the structure space \( U \) of the ring \( \mathfrak{A}_1 \). Let \( A \) be a subset of \( S'_1 \) and let \( \mathfrak{C} \in \mathfrak{A} \wedge S'_1 \) so that \( \mathfrak{C} \) is in the closure of \( A \) in the subspace \( S'_1 \). If \( \mathfrak{D}_A \) is the intersection of the primitive ideals in \( A \) then \( \mathfrak{C} \geq \mathfrak{D}_A \) but \( \mathfrak{C} \nsubseteq \mathfrak{A}_1 \). Let \( B \) be the subset of \( U \) of ideals \( \mathfrak{B} \wedge \mathfrak{A}_1 \) where \( \mathfrak{B} \in A \). The intersection of all of these ideals is the ideal \( \mathfrak{D}_A \wedge \mathfrak{A}_1 \). Since \( (\mathfrak{C} \wedge \mathfrak{A}_1) \geq (\mathfrak{D}_A \wedge \mathfrak{A}_1), \mathfrak{C} \wedge \mathfrak{A}_1 \) is in \( \mathfrak{B} \). Hence the mapping that we have defined between \( S'_1 \) and the subspace of \( U \) is a continuous one.

4. Let \( \{F_\alpha\} \) be a set of closed sets in the space \( S \). As before let \( \mathfrak{D}_{F_\alpha} \) denote the two-sided ideal of elements common to all the \( \mathfrak{B} \in F_\alpha \). Suppose that the intersection \( \Pi F_\alpha \neq \omega \) and let \( \mathfrak{B} \) be a point in this intersection. Then \( \mathfrak{B} \geq \mathfrak{D}_{F_\alpha} \) for all \( \alpha \). Hence \( \mathfrak{B} \) contains the two-sided ideal \( \Sigma \mathfrak{D}_{F_\alpha} \) generated by the \( \mathfrak{D}_{F_\alpha} \). The converse follows by retracing the steps of this argument. We therefore have the

**Lemma 1.** If \( \{F_\alpha\} \) is a set of closed sets in \( S \), \( \Pi F_\alpha \neq \omega \) if and only if \( \Sigma \mathfrak{D}_{F_\alpha} \) can be imbedded in a primitive ideal.

If \( \mathfrak{A} \) is a ring with an identity any two-sided ideal \( \neq \mathfrak{A} \) can be imbedded in a primitive ideal. Hence we have the

**Corollary.** If \( \mathfrak{A} \) is a ring with an identity and \( \{F_\alpha\} \) is a set of closed sets in \( S \) then \( \Pi F_\alpha \neq \omega \) if and only if \( \Sigma \mathfrak{D}_{F_\alpha} \neq \mathfrak{A} \).

If \( \mathfrak{A} \) has an identity and \( \Sigma \mathfrak{D}_{F_\alpha} = \mathfrak{A} \), \( 1 = d_1 + \ldots + d_s \) where \( d_i \in \mathfrak{D}_{F_i} \). Hence also \( \Sigma \mathfrak{D}_{F_i} = \mathfrak{A} \). The corollary therefore shows that if \( \Pi F_\alpha = \omega \)
then there is a finite set of closed sets $F_i$ such that $\bigcap F_i = \omega$. We therefore have the following

**Theorem 1.** The structure space of a ring with an identity is bicom pact.

Suppose now that $\mathfrak{A}$ is a semi-simple ring. Let $\mathfrak{A}$ be decomposable as a direct sum $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ of the two-sided ideals $\mathfrak{A}_i \not= 0$. Let $S_i$ be the closed subset of $S$ consisting of the primitive ideals $\mathfrak{B}$ containing the ideal $\mathfrak{A}_i$. Since $\mathfrak{A} - \mathfrak{A}_1 \cong \mathfrak{A}_2$ a semi-simple ring, $S_1 \not= \omega$. Similarly $S_2 \not= \omega$. Also the semi-simplicity of $\mathfrak{A} - \mathfrak{A}_1$ implies that $\mathfrak{A}_1$ is the intersection of all the $\mathfrak{B}$ in $S_i$. Since $\mathfrak{A}_1 + \mathfrak{A}_2 = \mathfrak{A}$ the lemma implies that $S_1 \cap S_2 = \omega$. Now let $\mathfrak{B}$ be any primitive ideal in $\mathfrak{A}$. We assert that either $\mathfrak{B} \supseteq \mathfrak{A}_1$ or $\mathfrak{B} \supseteq \mathfrak{A}_2$. For otherwise $\mathfrak{A}_1 + \mathfrak{B} > \mathfrak{B}$. The ideals $(\mathfrak{A}_1 + \mathfrak{B}) - \mathfrak{B}$ are $\not= 0$ in the primitive ring $\mathfrak{A} - \mathfrak{B}$. Since $[(\mathfrak{A}_1 + \mathfrak{B}) - \mathfrak{B}][(\mathfrak{A}_2 + \mathfrak{B}) - \mathfrak{B}] = 0$ this is impossible. We have therefore proved our assertion. Evidently it is equivalent to the relation $S_1 \lor S_2 = S$. Thus $S$ is disconnected into the two components $S_1$ and $S_2$.

Conversely suppose that $S = S_1 \lor S_2$ where the $S_i$ are closed sets $\not= \omega$ such that $S_1 \cap S_2 = \omega$. We assume also that $\mathfrak{A}$ has an identity. Let $\mathfrak{A}_{1} = \Delta S_i$. Then $\mathfrak{A}_{1} + \mathfrak{A}_{2} = \mathfrak{A}$. Since $\mathfrak{A}$ is semi-simple $\mathfrak{A}_{1} \cap \mathfrak{A}_{2} = \Delta S = 0$. If $\mathfrak{A}_{1} = 0, S = S_i = \omega$ contrary to $S_1 \not= \omega$ and $S_2 \not= \omega$. A part of our result is the following

**Theorem 2.** If $\mathfrak{A}$ is a semi-simple ring with an identity, $\mathfrak{A} = \mathfrak{A}_{1} \oplus \mathfrak{A}_{2}$ where the $\mathfrak{A}_i$ are two-sided ideals $\not= 0$ if and only if $S$ is not connected.

5. If $S'$ is a completely regular bicom pact space and $\mathcal{E}$ is the ring of real-valued (complex valued) continuous functions on $S'$ then it has been shown by Gelfand and Silov that the structure space $S$ of $\mathcal{E}$ is homeomorphic to $S'$. If $S'$ is a bicom pact totally disconnected space and $\mathcal{E}$ is the ring of continuous functions on $S'$ having values in the field of residues mod 2, then by a result of Stone's the structure space of $\mathcal{E}$ is homeomorphic to $S'$.

We conclude this note by giving another example of this type based on an arbitrary totally disconnected bicom pact space $S'$ and an arbitrary division ring $\mathcal{R}'$.

We consider any decomposition of $S'$ into a finite number of components (non-overlapping open and closed sets) $S'_i$ and we choose corresponding elements $k_i \in \mathcal{R}'$. We define a function $f(x)$ by setting $f(x_i) = k_i$ for $x_i$ in $S'_i$. A function of this type will be called a finite decomposition function. The totality $\mathcal{E}$ of these functions is a ring under the ordinary operations of addition and multiplication. $\mathcal{E}$ contains the subring $\mathcal{R}$ of constant functions, isomorphic to $\mathcal{R}'$ and $\mathcal{E}$ is commutative if and only if $\mathcal{R}'$ is commutative. The constant 1 acts as an identity in $\mathcal{E}$. Since $S'$ is totally disconnected, for any two points $a \not= b$ in $S'$ there is an $f(x) \in \mathcal{E}$ such that $f(a) \not= f(b)$. Let $\mathfrak{A}$ be any subring of $\mathcal{E}$ having this property and containing $\mathcal{R}$. We shall sketch a proof of the fact that the space $M$ of maximal two-sided ideals of $\mathfrak{A}$ is homeomorphic to $S'$. 
Lemma 2. If $F'$ is a closed subset of $S'$ and $a$ is a point in $F'$ then there exists a function $\varphi(x) \in \mathfrak{A}$ such that $\varphi(y) = 0$ for all $y \in F'$ but $(\varphi a) \neq 0$.

Our assumptions imply that for each $y \in F'$ there is an $f_y \in \mathfrak{A}$ such that $f_y(y) = 0$ but $f_y(a) \neq 0$. The set $Z'(f_y)$ of zeros of $f_y$ is open and the totality of these sets covers $F'$. Let $Z'(f_{y_1}), \ldots, Z'(f_{y_m})$ be a finite subset of these sets covering $F'$. Then $\varphi(x) = f_{y_1}(x) \ldots f_{y_m}(x)$ has the required properties.

If $a \in S'$ we let $\mathfrak{B}_a$ denote the totality of functions $g(x) \in \mathfrak{A}$ such that $g(a) = 0$. It is easy to see that $\mathfrak{B}_a$ is a maximal two-sided ideal in $\mathfrak{A}$ and that $\mathfrak{A} - \mathfrak{B}_a \cong \mathfrak{B}_a$. Our conditions imply that if $a \neq b$ then $\mathfrak{B}_a \neq \mathfrak{B}_b$.

Lemma 3. If $\mathfrak{B}$ is a two-sided ideal $\neq \mathfrak{A}$, then there exists a point $a$ such that $g(a) = 0$ for all $g \in \mathfrak{B}$.

Let $Z'(f)$ be the set of zeros of $f$. $Z'(f)$ is closed. If the lemma is false, the intersection $\Pi Z'(f)$ for all $f \in \mathfrak{B}$ is vacuous. Hence there is a finite, number of functions $f_1, \ldots, f_n$ in $\mathfrak{B}$ such that $\Pi Z'(f_i) = \omega$. Hence if $a$ is any point of $S'$ at least one of the functions $f_i$ does not vanish at $a$. Let $k_{n_1}, \ldots, k_{n_n}$ be the non-zero values taken on by $f_i$ and form the function $\psi_i(x) = (f_i(x) - k_{n_1})(f_i(x) - k_{n_2}) \ldots (f_i(x) - k_{n_n})$ and $\psi(x) = \psi_1(x)\psi_2(x) \ldots \psi_r(x)$. Then $\psi(x) = 0$. The form of $\psi(x)$ shows that $0 = \psi(x) = k + g(x)$ where $k \neq 0$ and $g(x) \in \mathfrak{B}$. Hence $\mathfrak{B}$ contains a constant function $\neq 0$ and $\mathfrak{B} = \mathfrak{A}$ contrary to assumption.

This lemma shows that $\mathfrak{B} \leq \mathfrak{B}_a$ for some $a$. If $\mathfrak{B}$ is maximal $\mathfrak{B} = \mathfrak{B}_a$. Thus the correspondence $a \mapsto \mathfrak{B}_a$ is $(1 - 1)$ between $S'$ and the space $M$ of maximal two-sided ideals in $\mathfrak{A}$.

Let $F'$ be a closed subset of $S'$ and let $F$ be the corresponding set in $M$. The intersection $\mathfrak{D}_F = \Pi \mathfrak{B}_a$ for all $a \in F'$ is the set of functions $f$ such that $f(a) = 0$ for all $a \in F$. Let $\mathfrak{B}_b$ be a maximal two-sided ideal containing $\mathfrak{D}_F$. If $b \in F'$ there is a function $\phi$ such that $\phi (a) = 0$ for all $a \in F'$ but $\phi (b) \neq 0$. Then $\phi \in \mathfrak{D}_F$ but $\phi \in \mathfrak{B}_b$ contrary to $\mathfrak{B}_b \geq \mathfrak{D}_F$. Hence $b \in F'$ and $\mathfrak{B}_b \in F$. Thus $F$ is closed. Conversely let $F$ be any closed set in $M$ and let $F'$ be the corresponding set in $S'$. It is easy to see that $F'$ is the set of points $y$ such that $f(y) = 0$ for all $f \in \mathfrak{D}_F$. Thus $F'$ is the intersection of closed sets and is therefore closed. Hence the correspondence $a \mapsto \mathfrak{B}_a$ is a homeomorphism.

Theorem 3. Let $S'$ be a totally disconnected bicompact space, $\mathfrak{R}$ a division ring and $\mathfrak{A}$ a ring of $\mathfrak{R}$-valued decomposition functions such that (1) $\mathfrak{A}$ contains the constants and (2) if $a \neq b$ in $S'$, then $\mathfrak{A}$ contains a function $f$ such that $f(a) \neq f(b)$. Then the space $M$ of maximal two-sided ideals of $\mathfrak{A}$ is homeomorphic to $S'$.

If $\mathfrak{R}$ is commutative, $M = S$. It is an open question whether or not this is true in general. However, it is clear that $\mathfrak{M} = S$ since the intersection of the set of maximal two-sided ideals is (0).

Theorem 3 shows that two rings $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of the type considered are not isomorphic unless the spaces $S_1'$ and $S_2'$ are homeomorphic. Since the division ring $\mathfrak{R}_t'$ is isomorphic to $\mathfrak{A}_t - \mathfrak{B}_t$ for any maximal two-sided ideal
\[ \mathfrak{A}_4 \text{ in } \mathfrak{A}_4, \text{ the isomorphism of } \mathfrak{A}_1 \text{ and } \mathfrak{A}_2 \text{ implies that of } \mathfrak{A}_1' \text{ and } \mathfrak{A}_2'. \] If 
\[ \mathfrak{A}_4 = \mathfrak{S}_t \text{ the complete ring of finite decomposition functions then the converse holds.} \text{ Hence necessary and sufficient conditions that } \mathfrak{S}_1 \text{ and } \mathfrak{S}_2 \text{ be isomorphic are that } S_1' \text{ and } S_2' \text{ be homeomorphic and that } \mathfrak{R}_1' \text{ and } \mathfrak{R}_2' \text{ be isomorphic.} \]

1 "The Radical and Semi-simplicity for Arbitrary Rings," *Amer. Jour. Math.*, 67, 300-320 (1945). The results that we state without proof in this section can be found in the above-mentioned paper.

2 We use the notations \( \lor \) and \( \land \), respectively, for the logical sum and logical product of a finite number of sets.


5 Loc. cit. in reference 1, p. 313.


7 Loc. cit. in reference 1, p. 313.

8 Loc. cit. in reference 4, p. 380. It is assumed here that the field of residues mod 2 is endowed with a \( T_1 \)-topology. This means that each point is both open and closed.

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**CONVERSE OF PTOLEMY’S THEOREM ON STEREOGRAPHIC PROJECTION**

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1. **The Problem.**—The famous theorem of Ptolemy on stereographic projection states that the perspectivity of a sphere from a point of the sphere upon a plane perpendicular to the diameter of the sphere determined by the given point is conformal. The question naturally arises as to the existence of any other surfaces for which there exists a perspectivity upon a plane which is conformal. We prove that there does not exist any other surface with this property (except for the obvious case of a parallel plane). Our result may be stated as follows:

**Theorem.** *If a perspectivity of a surface from a fixed point to a fixed plane is conformal, then the surface must be a sphere; furthermore, the sphere must pass through the fixed point and have its center on the perpendicular from the fixed point to the fixed plane.*

Thus the only perspective conformalities upon a plane are Ptolemy’s stereographic projection of the sphere (or the limiting case of the plane).

2. **Beginning of the Proof of the Theorem.**—Let \( (x, y, z) \) denote cartesian