3. These two problems make somewhat heavier demands than the classical convergence questions on the methods of functional analysis. By the Banach-Steinhaus theorem the problems can be reduced to showing that the norms (in the appropriate space) of the partial sums are bounded, i.e., if \( s_n(f) \) denotes the partial sums of the series corresponding to \( f(x) \), then \( \|s_n(f)\| \leq M\|f\| \) for some positive \( M \) and all \( f \) in the space. This is accomplished by two devices, of which the first alone suffices for trigonometrical series: M. Riesz’ theory of conjugate functions, and the following inequality.

**Lemma.** If \(-1 < c < 1, c < \frac{1}{p} < c + 1, p > 1, \) and \( f(x) \) belongs to \( L^p(-1, 1) \), then so does

\[
g(x) = \int_{-1}^{1} \left| \frac{1 - y^p}{1 - x^2} \right|^c \frac{f(y)dy}{x - y}.
\]

Moreover \( \|g\|_p \leq A\|f\|_p \), where \( A \) is independent of \( f \).

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**THE FUNDAMENTAL THEOREM ON QUADRATIC FIRST INTEGRALS**

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In the following we have shown that the set of all homogeneous quadratic first integrals of the differential equations of the paths of an affinely connected space admits a finite basis. The demonstration is so devised that the number of integrals in the basis is identical with the number of solutions in a fundamental system of solutions of a certain set of linear homogeneous equations; hence the determination of this number is reduced to the solution of an algebraic problem.

A corresponding basis theorem holds for the integrals of energy type of a conservative dynamical system. Due to the importance of such integrals for the dynamical problem the proof in question has been indicated and the result stated in the form of a theorem.
It suffices for the demonstration to assume the analyticity of the various functions involved. But this assumption is far too drastic. The differentiability requirements should be such as to permit the construction of the above-mentioned system of linear homogeneous equations by which the number of integrals in the basis is determined. In general, however, the process by which this system is obtained will speedily terminate yielding moderate requirements of differentiability.

**Integrals in General Affine Space.**—Consider the conditions under which the differential equations

\[
\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0
\]

defining the paths of a general affinely connected space of symmetric affine connection \( \Gamma \), admit a homogeneous quadratic integral

\[
g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \text{const.} \tag{2}
\]

Among these various conditions we signalize the following as the basis of our discussion. First, there exist relations of the form

\[
\delta_{ij,p,q} = \delta_{ij,p} - \varepsilon_{ij} A^\alpha_{ijpq} - \varepsilon_{ij} A^\alpha_{ijpq} \tag{3}
\]

\[
g_{ij,p,q} = \varepsilon_{ij} E^\alpha_{ijpq} + \varepsilon_{ij} F_{ijpq} \tag{4}
\]

where the \( \varepsilon_{ij,\gamma} \) and the \( \delta_{ij,p,q} \) are the components of the first covariant derivative (first extension) and second extension, respectively, of the tensor defined by the coefficients in (2), etc.; also the quantities \( A^\alpha_{ijpq} \) are the components of the first normal tensor. System (3) is an identity in the space. System (4) is satisfied in virtue of the fact that (2) is a quadratic integral. Second, there exists a sequence of sets of equations

\[
S_0 = 0, \ S_1 = 0, \ S_2 = 0, \ldots \tag{5}
\]

each of which is linear and homogeneous in the quantities \( g_{ij} \), \( g_{ij,p} \) and \( g_{ij,p,q} \) with coefficients which are constants or tensor invariants of the space. A necessary and sufficient condition for the existence of an integral (2) is that there is an integer \( N \) such that the first \( N + 1 \) sets of equations of the sequence (5) are consistent as equations for the determination of the quantities \( g_{ij} \), \( g_{ij,p} \) and \( g_{ij,p,q} \) and that all their solutions satisfy the \( (N + 2)nd \) set of equations of the sequence. Third, if \( g_{ij}^\alpha \), \( g_{ij,p}^\alpha \) and \( g_{ij,p,q}^\alpha \) where \( \alpha = 1, \ldots, s \) and \( s \geq 1 \) is a fundamental system of solutions of the first \( N + 1 \) sets of the sequence, the general solution of these equations is given by

\[
g_{ij} = \phi^\alpha g_{ij}^\alpha, \ g_{ij,p} = \phi^\alpha g_{ij,p}^\alpha, \ g_{ij,p,q} = \phi^\alpha g_{ij,p,q}^\alpha \tag{6}
\]

where \( \alpha \) is summed over the values 1, \ldots, \( s \) and the \( \phi^\alpha \)'s are arbitrary func-
tions of the \( x \)'s. A sufficient condition for (2) to be an integral is now that 
\[ g_{ij} = \phi^a g^a_{ij} \]
where the \( \phi \)'s satisfy a certain completely integrable system of the form

\[
\frac{\partial \phi^b}{\partial x^a} + \phi^a \lambda_{b}^a = 0. \tag{7}
\]

Assuming the above conditions for the existence of the integral (2) to be satisfied we can now determine \( s \) sets of functions \( \phi^a \) as solutions of (7) by the following \( s \) sets of initial conditions

\[ (1, 0, 0, \ldots, 0), \ (0, 1, 0, \ldots, 0), \ldots, \ (0, 0, 0, \ldots, 1). \]

We thus obtain \( s \) integrals (2) with \( g \)'s given by \( g^a_{ij} \) where \( \alpha = 1, \ldots, s \). Initially we have

\[ g_{ij}^a = g_{ij}^a, \ g_{ij,\phi}^a = g_{ij,\phi}^a, \ g_{ij,\phi q}^a = g_{ij,\phi q}^a \]

where the quantities on the left are the components of the tensors \( g^a \) and their first and second extensions while the quantities on the right are the above fundamental system of algebraic solutions. Hence these left-hand quantities can be taken as a fundamental system of solutions of the first \( N + 1 \) sets of equations (5). We state this result as the following theorem.

**Theorem 1.** If the first \( N + 1 \) sets of equations (5) admit a fundamental system of \( s (\geq 1) \) algebraic solutions each of which satisfies the \( N + 2 \)nd set of these equations, then there exists \( s \) quadratic integrals

\[
g^a_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \text{const.,} \ (\alpha = 1, \ldots, s), \tag{8}
\]

such that the tensors \( g^a \), which are determined by the coefficients of these integrals, yield a fundamental system of solutions \( g^a_{ij}, \ g^a_{ij,\phi}, \ g^a_{ij,\phi q} \) of the first \( N + 1 \) sets of the equations. We use this fundamental system of solutions in the following discussion.

The \( s \) integrals (8) are linearly independent in the sense that the equations \( c_a g^a_{ij} = 0 \), in which the \( c \)'s are constants, imply that \( c_a = 0 \). For, it follows from these equations that \( c_a g^a_{ij,\phi} = 0 \) and \( c_a g^a_{ij,\phi q} = 0 \); hence all \( c_a = 0 \) since otherwise we would have a linear relation between the solutions of our fundamental system.

Now let (2) be any quadratic integral. Then

\[
g_{ij} = \phi^a g^a_{ij}, \ g_{ij,\phi} = \phi^a g^a_{ij,\phi}, \ g_{ij,\phi q} = \phi^a g^a_{ij,\phi q} \tag{9}
\]

for suitable \( \phi \)'s. Hence from (3), (4) and (9) we have

\[
(\phi^a g^a_{ij})_{,\phi} = \phi^a g^a_{ij,\phi} \\
(\phi^a g^a_{ij,\phi})_{,\phi} = \phi^a g^a_{ij,\phi q} - \phi^a g^a_{ij} A^a_{ij,\phi q} - \phi^a g^a_{ij} A^a_{ij,\phi q} \\
(\phi^a g^a_{ij,\phi q})_{,\phi} = \phi^a g^a_{ij} A^a_{ij,\phi q} + \phi^a g^a_{ij} A^a_{ij,\phi q}.
\]
Performing the indicated differentiations in the left members of these equations, and using the fact that the quantities $g'_{ij}$ (for each value of $\alpha$) satisfy (3) and (4), it follows that

$$\phi_{\alpha}^\alpha g_{ij}^\alpha = 0, \quad \phi_{\alpha}^\alpha g_{ij,\alpha} = 0, \quad \phi_{\alpha}^\alpha g_{ij,\alpha\beta} = 0. \quad (10)$$

But (10) implies that $\phi_{\alpha}^\alpha = 0$ since otherwise there would exist a linear relation between the solutions of the fundamental system. Hence $\phi_{\alpha}^\alpha = \text{const}$. Hence,

**Theorem II.** If quadratic integrals (2) of the differential equations (1) exist, then the $s$ integrals (8) constitute a basis for the set of all such integrals, i.e., if (2) is any quadratic integral we will have $g_{ij} = c_{ij}^\alpha$ where the $c$'s are constants.

Remark 1.—Similar theorems can be proved for linear first integrals of (1) on the basis of results (loc. cit., p. 591) analogous to those used in the proof of the above theorems for quadratic integrals. The methods are general and one can conclude that corresponding theorems are true for homogeneous first integrals of any degree.

Remark 2.—The above theorems on quadratic integrals apply in particular when (1) are the differential equations of the geodesics of a Riemann space. In this case the $I'$s are the well-known Christoffel symbols and the basis of quadratic integrals can be considered to contain the quadratic integral determined by the fundamental metric tensor of the space.

Remark 3.—For the particular class of integrals

$$a_{\alpha\beta\ldots\gamma} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \ldots \frac{dx^\gamma}{ds} = \text{const}. \quad (11)$$

such that $a_{\alpha\beta\ldots\gamma,\delta} = 0$ the sequence corresponding to (5) is readily constructed (loc. cit., p. 558) and yields results of the type expressed by the above Theorems I and II. The quadratic integrals of this class are of especial interest from the geometrical standpoint.

**Quadratic First Integrals in Flat Space.**—If the space is flat the equations $S_k = 0$ for $k = 1, 2, \ldots$ are satisfied identically and the set of equations $S_0 = 0$ becomes

$$g_{ij,p} + g_{jp,i} + g_{pi,t} = 0, \quad g_{ij,pq} + g_{jp,q} + g_{pi,t} = 0,$$

$$g_{ij} = g_{ij,p} + g_{ij,p} = g_{ij,t}, \quad g_{ij,pq} = g_{ij,p} + g_{ij,q} = g_{ij,t}.$$

Hence we can take $N = 1$ in Theorem I and the number $s$ of quadratic integrals (8) in the basis is equal to the number of algebraically independent quantities $g_{ij}, g_{ij,p}$ and $g_{ij,pq}$ in the above equations. We thus find

$$s = \frac{n(n + 1)^2(n + 2)}{12}. \quad (12)$$

Similar remarks apply to integrals of the type (11); we note in particular
that \( n(n + 1)/2 \) is the number of linearly independent quadratic integrals of this type.

It is evident that the number \( s \) given by (12) is an upper bound to the number of quadratic integrals in the basis for any system (1). Similarly, \( n(n + 1)/2 \) is an upper bound for the number of quadratic integrals of type (11), i.e., which satisfy the condition \( a_{\alpha \beta, \gamma} = 0 \). The above discussion shows that in the case of a flat space these upper bounds are actually attained.

We leave open the question of whether the existence of this maximum number of integrals is a sufficient condition for the space to be flat.

**Quadratic Integrals of a Dynamical System.**—The differential equations of the trajectories of a conservative dynamical system can be written

\[
\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = -g^\alpha_{\beta} V^\beta,
\]

where \( V(x^1, \ldots, x^n) \) is the potential and the \( \Gamma \)‘s are Christoffel symbols based on the coefficients \( g_{\mu \nu} \) of the quadratic form

\[
T = \frac{1}{2} g_{\mu \nu}(x) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}
\]

defining the kinetic energy \( T \). We assume that the \( g_{\mu \nu} \) do not involve the time \( t \) explicitly. The system (13) then admits the quadratic integral

\[
\frac{1}{2} g_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + V = \text{const.}
\]

which expresses the condition that the sum of kinetic and potential energies is constant along any trajectory.

It should be possible to prove theorems analogous to Theorems I and II for integrals of the type (15). Here, however, we prove only the finite basis theorem for such integrals as a consequence of Theorem II. Thus consider any integral of (13) of the type (15), namely,

\[
\frac{1}{2} h_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + W = \text{const.}
\]

The differential conditions on the quantities \( h_{\mu \nu} \) and \( W \) for (16) to be an integral of (13) are readily deduced and indicate that

\[
h_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = \text{const.}
\]

is an integral of the differential equations of the geodesics, i.e., the equations obtained from (13) by replacing the right members by zero. Hence, dealing only with those quadratic integrals (17) of the equations of geodesics which are associated with integrals (16) of (13) it now follows from the
above basis theorem for homogeneous quadratic first integrals that there exists a finite number of these integrals (17) in terms of which any other such integral can be expressed linearly with constant coefficients. In other words there exist \( s(\geq 1) \) integrals

\[
\frac{1}{2} \sum_{\mu \nu} \varepsilon_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + V^\alpha = \text{const.,} \quad (\alpha = 1, \ldots, s),
\]

(18)
of (13) such that if (16) is any integral of (13) we must have

\[
h_{\mu \nu} = c_{a\nu \rho},
\]

(19)
where the \( c \)'s are constants. Now

\[
\frac{1}{2} \sum_{\nu \rho} c_{a\nu \rho} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + c_\alpha V^\alpha = \text{const.}
\]

(20)
is an integral of (13) for any selection of the constants \( c_\alpha \). But if we choose the \( c \)'s to satisfy (19) it follows from (16) and (20) that \( W - c_\alpha V^\alpha = \text{const.} \) along any trajectory. Differentiating this equation with respect to \( t \) we have \( (W - c_\alpha V^\alpha) \frac{dx^\rho}{dt} = 0 \) along the trajectory and since the \( dx^\rho/dt \) can be chosen arbitrarily it follows that \( (W - c_\alpha V^\alpha)_\beta = 0 \). Hence

\[
W = c_\alpha V^\alpha + k,
\]

(21)
where \( k \) is an absolute constant.

We can suppose that there exists no linear relation with constant coefficients, not all of which are zero, between the left members of (18) since otherwise the number \( s \) of these equations could be reduced. The set of integrals (18) is then said to be linearly independent. We state the above result as the following theorem.

**Theorem III.** There exists a finite number \( s(\geq 1) \) of linearly independent integrals (18) of the conservative dynamical system (13) such that any other integral (16) is determined by the relations (19) and (21) in which the \( c \)'s and \( k \) are constants.

The set of linearly independent integrals (18) occurring in the above theorem will be said to form a basis of integrals for the dynamical system.