CONVERSE THEORY OF GNOMIC AND EQUIAREAL PERSPECTIVITIES*

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1. Perspective Conformalities.—We shall present some theorems on the perspective mapping of a surface upon a plane from a given point. In an earlier paper, we have proved the following result.

The only perspective conformalities upon a plane are Ptolemy's stereographic projection of a sphere (and the obvious limiting case of a parallel plane).

Thus there are no surfaces except for spheres and planes, for which there exists a perspectivity upon a plane from a given point, which is conformal.

In our present work, we shall consider some properties of gnomic projection of a sphere upon a plane (geodesic mapping). Also, we shall study surfaces for which area-preserving perspectivities exist.

2. Perspective Representation of Geodesics.—Among Kasner's theorems on the problem of partial geodesic representation and the near-collineation problem, may be found the following proposition. If there exists a point-to-point representation of a surface upon a plane such that more than 3 co geodesics correspond to straight lines, then all geodesics correspond to straight lines. Therefore, by a theorem of Beltrami the surface is of constant curvature.

We shall restrict our point transformations to perspectivities. The condition of constant curvature for our case is only necessary but not sufficient. We shall discuss the following theorem.

Characterization of gnomic projection. If more than 3 co geodesics are projected into straight lines under a perspectivity, then all geodesics project into straight lines, and the surface is a sphere (or the obvious case of any plane, parallel or not); furthermore the point of perspectivity is at the center of the sphere.

We shall deduce from our work the following classification of surfaces according to the number of geodesics which are projected into straight lines by a perspectivity: There are four distinct classes.

(I) The non-ruled surface. At most 1 co.
(II) The ruled surfaces excluding the quadrics. There are always 1 co (the rulings) and at most 2 co.
(III) The quadrics excluding the gnomic projections of spheres. There are always 2 co (the two systems of rulings) and at most 3 co.
(IV) The gnomic projection of a sphere, and the limiting case of any plane. All \( \approx 2 \).

5. Surfaces with Plane Geodesics.—Elsewhere\(^5\) we have proved the following result which is closely related to the above.

The spheres are the only surfaces which possess more than \( 2^\infty \) non-rectilinear plane geodesics.

The discussion of surfaces with reference to the maximum number of plane geodesics, straight or not, leads to the same classification as above. The maximum number of plane geodesics for each class is as follows.

(I) At most \( 2^\infty \).

(II) Always \( \approx 1 \) and at most \( 3^\infty \).

(III) Always \( 2^\infty \) and at most \( 4^\infty \).

(IV) All \( \approx 2 \).

4. Discussion of Our Characterization of Gnomic Projection.—Let \((x, y, z)\) denote cartesian coördinates of a point. Let \( z = f(x, y) \) be the equation of our surface \( \Sigma \). Introduce the usual notation: \( p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}; r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2} \).

Take the origin as the center of perspectivity and \( z = c \neq 0 \) as the given plane \( \pi \). The perspectivity from the given point \( 0(0, 0, 0) \) upon the plane \( \pi:(X, Y, c) \) of the surface \( \Sigma:(x, y, z = f(x, y)) \), is

\[
X = cx/z, \quad Y = cy/z. \quad (1)
\]

The jacobian \( j \) of this perspectivity is

\[
j = \frac{c^2}{z^2} (z - xp - yq) \neq 0. \quad (2)
\]

Observe that the surface \( \Sigma \) cannot be a plane through the origin, or a cone with vertex at the origin.

It is found by (1) that the differential equation defining the straight lines of the plane \( \pi \), is

\[
(z - xp - yq)y'' = (y - xy')(r + 2sy' + ty''). \quad (3)
\]

The geodesics of the surface \( \Sigma \) are given by the differential equation

\[
(1 + p^2 + q^2)y'' = (-q + p \cdot y')(r + 2sy' + ty''). \quad (4)
\]

Eliminating \( y'' \) from (3) and (4), we find that the possible geodesics which are projected into straight lines satisfy either

\[
r + 2sy' + ty'' = 0, \quad (5)
\]

or

\[
[(1 + q^2)(xp + x) - pq(xq + y)]y' = [-pq(xp + x) + (1 + p^2)(xq + y)]. \quad (6)
\]
For any surface where all the geodesics are projected into straight lines, either (5) or (6) or both, are identities in $y'$. These identities will demonstrate that any such surface $\Sigma$ is either a plane in general position, or a sphere with center at the origin. This result can be deduced also from the theorem stated in Section 3.

The classification of surfaces with reference to the number of geodesics which are projected into straight lines, may be deduced from the following observations. Firstly, if (5) represents an infinitude of geodesics, these must be straight lines and hence the surface is ruled. Secondly, the quadrics are the only ruled surfaces with two distinct systems of rulings. Thirdly and finally, the non-rectilinear geodesics which project into straight lines must satisfy (6).

5. Surfaces for Which Area-Preserving Perspectivities Exist.—By (2), the area formula in the plane $\pi$, is

$$\text{Area in } \pi = c^2 \int \int \frac{1}{z^4} (z - xp - yg) dx dy. \quad (7)$$

The area formula in the surface $\Sigma$ is

$$\text{Area in } \Sigma = \int \int (1 + p^2 + q^2)^{1/2} dx dy. \quad (8)$$

Therefore, all surfaces for which area-preserving perspectivities exist, must satisfy the partial differential equation of first order

$$c^4(z - xp - yg)^2 - z^6(1 + p^2 + q^2) = 0. \quad (9)$$

All plane solutions of this equation are $z = 0$ (this is the singular solution), $z = \pm c$, and $z = \pm ic$. Thus the only real non-trivial plane surface for which an area-preserving perspectivity exists, is the plane $\pi': z = -c$.

It can be proved that there are no spherical surfaces which satisfy the partial differential equation (9); so equiareal perspectivity of the sphere is impossible.

In order to study this partial differential equation of first order (9), in the real domain, it is found convenient to introduce the following algebraic surface $R$ of revolution of the sixth degree. It is the locus of a point such that its distance from the point of perspectivity $0(0, 0, 0)$ is equal to the ratio of the cube of its distance from the plane $\pi_0: z = 0$, to the square of the distance $c$ of the plane $\pi: z = c$, from $\pi_0$. The equation of this algebraic surface $R$ is

$$x^2 + y^2 + z^2 = z^6/c^4. \quad (10)$$

This surface $R$ has an isolated singularity at the point of perspectivity 0, the tangent directions being on the minimal cone with vertex at 0. Otherwise the surface $R$ is defined for $z \geq c$ and $z \leq -c$. 


Note that the partial differential equation (9) may be written in the form
\[(y + qz)^2 + (x + pz)^2 + (yp - xq)^2 - \left( x^2 + y^2 + z^2 - \frac{z^2}{c^4} \right) \times \]
\[\left( 1 + p^2 + q^2 \right) = 0. \tag{11} \]
For real solutions we must have \[x^2 + y^2 + z^2 \geq \frac{z^2}{c^4} \].

In the real domain, the partial differential equation (9) is defined only in the region bounded by the algebraic surface \(R\) of sixth degree, which contains the point of perspectivity 0. At any point \(P\) of \(R\), there is only one real planar direction whose normal is the radius vector of \(P\).

It is found that at each point \((x, y, z)\), the surface elements of (9), envelope the quadric cone
\[(x dx + y dy + z dz)^2 - (x^2 + y^2 + z^2 - \frac{z^2}{c^4})(dx^2 + dy^2 + dz^2) = 0. \tag{12} \]
These quadric cones can degenerate only along our algebraic surface \(R\).

By Charpit's method, the complete integrals of the partial differential equation of first order (9), are cylinders with elements parallel and symmetrical to a line through the point of perspectivity, all of which are parallel to the plane \(\pi\). The directorial curve is expressed parametrically in terms of a Tchebycheff integral of non-elementary character.

The axis of the cylinders may be taken as the \(y\)-axis. The directorial curve is in the \(xz\)-plane and its differential equation can be written in the form
\[x = z \frac{dx}{dz} = \frac{z^2}{c^4} \left[ 1 + \left( \frac{dx}{dz} \right)^2 \right]^{\frac{1}{2}}. \tag{13} \]

Taking the inclination \(t\) of the tangent line of this curve to the \(z\)-axis as parameter, it is found that this curve is given by the parametric equations
\[x = z \tan t = \frac{z^2}{c^2} \sec t, \quad z^2 = \frac{2c^2}{3} \cos^{\frac{1}{2}} t[K = \int \sec^{\frac{1}{2}} tdt], \tag{14} \]
where \(K\) is a constant of integration. The integral which appears above may be shown to be reducible to a Tchebycheff integral. By a theorem of Liouville, this is found to be not expressible in terms of the elementary functions.

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1 Kasner and De Cicco, "Converse of Ptolemy's Theorem on Stereographic Projection," these PROCEEDINGS, 31, 338-342 (1945).
