of $E^k$, $R$ is the $E^n$ and the mapping $x(p)$ satisfies a Lipschitz condition $x(p)x(q) < \rho \cdot pq$, then $\alpha(S) = \alpha_k(S)$ coincides with the classical integral and, therefore, with the various other areas defined by Peano Lebesgue, Radô and Federer. For non-rectifiable surfaces the relations are mostly unclear.

Proofs will be forthcoming in a comprehensive paper with the same title.

3 Hurewicz, Witold, and Wallman, Henry, Dimension Theory, Princeton, 1941.

\[ \lim_{N \to \infty} \sum_{n=0}^{N} a_n p_n(x)^2 \text{ satisfies a Lipschitz condition}. \]

\[ \lim_{N \to \infty} \sum_{n=0}^{N} a_n p_n(x)^2 = 0. \]
In the language of functional analysis: for what values of $p$ do the orthonormal polynomials associated with $F(x)$ form a basis for $L^p$?

**Theorem 1.** If $F'(x) = (1 - x^2)\lambda^{-1/2}$, $-1 < x < 1$, $\lambda \geq 0$ then the associated (ultraspherical) polynomials form a basis for $L^p(-1, 1)$ if

$$2 - \frac{1}{\lambda + 1} < p < 2 + \frac{1}{\lambda}$$

but not if $1 \leq p < 2 - \frac{1}{\lambda + 1}$ or $p > 2 + \frac{1}{\lambda}$.

**Remarks on Theorem 1.**—(i) If $\lambda = 0$ the result follows from M. Riesz' theorem on the mean convergence of Fourier series. (ii) If $\lambda = \frac{1}{2}$ Theorem 1 establishes a conjecture (unpublished) of Zygmund that the Legendre polynomials form a basis for $L^p(-1, 1)$ if $\frac{4}{3} < p < 4$. (iii) The limiting case $\lambda = 0$ is interesting. If $x$ in $(1 - x^2)\lambda^{-1/2}$ is replaced by $x\lambda^{-1/2}$, and then the limit taken as $\lambda \to 0$, then the corresponding polynomials become those of Hermite on $(-\infty, \infty)$. The formula (3) suggests that for these polynomials mean-convergence holds only for $p = 2$. This, and a similar result for Laguerre polynomials, can be confirmed by suitable counter-examples. (iv) The end values $p = 2 - \frac{1}{\lambda + 1}$, $2 + \frac{1}{\lambda}$ are left open.

2. If $F(x)$ is absolutely continuous there is another interpretation of mean convergence. Let $F'(x) = w(x)$. Then the functions $\{w^{1/2}(x)p_n(x)\}$ form an orthonormal set in the ordinary sense; this follows from (1). With each function in $L^p(a, b)$, one can associate a series

$$f(x) \sim \sum b_n w^{1/2}(x)p_n(x),
\quad b_n = \int_a^b f(x)w^{1/2}(x)p_n(x)dx.$$

The question now is to determine when

$$\lim_{N \to \infty} \int_a^b |f(x) - \sum_0^N b_n w^{1/2}(x)p_n(x)|^p dx = 0.$$

In a sense this question is less "natural" than that of §1 for it is only by virtue of the accident that $F(x)$ is absolutely continuous that it has meaning. The following is true.

**Theorem 2.** If $w(x) = (1 - x)^\alpha (1 + x)^\beta$, $-1 < x < 1$, $\alpha \geq 0$, $\beta \geq 0$, then the functions $\{w^{1/2}(x)p_n(x)\}$ form a basis for $L^p(-1, 1)$ if $\frac{4}{3} < p < 4$, but not if $1 \leq p < \frac{4}{3}$, or $p > 4$.

**Remarks on Theorem 2.**—(i) The corresponding polynomials are, of course, those of Jacobi. (ii) It is of interest to observe that the range of $p$ is independent of the specific choice of $\alpha, \beta$. (iii) The first part of the theorem is true even if $\alpha \geq -\frac{1}{2}$, $\beta \geq -\frac{1}{2}$; the second part is open in this more general case. (iv) The end values $p = \frac{4}{3}, 4$ are open.
3. These two problems make somewhat heavier demands than the classical convergence questions on the methods of functional analysis. By the Banach-Steinhaus theorem the problems can be reduced to showing that the norms (in the appropriate space) of the partial sums are bounded, i.e., if \( s_n(f) \) denotes the partial sums of the series corresponding to \( f(x) \), then \( ||s_n(f)|| \leq M||f|| \) for some positive \( M \) and all \( f \) in the space. This is accomplished by two devices, of which the first alone suffices for trigonometrical series: M. Riesz' theory of conjugate functions, and the following inequality.

**Lemma.** If \(-1 < c < 1\), \( c < \frac{1}{p} < c + 1 \), \( p > 1 \), and \( f(x) \) belongs to \( L^p \) \((-1, 1)\), then so does

\[
g(x) = \int_{-1}^{1} \frac{\left(1 - y^2\right)^c}{\left(1 - x^2\right)} \frac{f(y)dy}{x-y}.
\]

Moreover \( ||g||_p \leq A||f||_p \), where \( A \) is independent of \( f \).

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1 See, for example p. 24 of G. Szegő's *Orthogonal Polynomials*, New York, 1939.
2 For this use of the word "basis" see S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.
4 Szegő, op. cit., Chapter IV.
5 Banach, op. cit., p. 79.

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**THE FUNDAMENTAL THEOREM ON QUADRATIC FIRST INTEGRALS**

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Communicated November 26, 1945

In the following we have shown that the set of all homogeneous quadratic first integrals of the differential equations of the paths of an affinely connected space admits a finite basis. The demonstration is so devised that the number of integrals in the basis is identical with the number of solutions in a fundamental system of solutions of a certain set of linear homogeneous equations; hence the determination of this number is reduced to the solution of an algebraic problem.

A corresponding basis theorem holds for the integrals of energy type of a conservative dynamical system. Due to the importance of such integrals for the dynamical problem the proof in question has been indicated and the result stated in the form of a theorem.