H(α) is h times a generator of π_{2r-1}(S^{2r-1}). Thus H may be regarded as a generalization of the Hopf invariant.

Let \( R_{r-1} \) be the rotation group of \( S^{r-1} \), \( π \) the mapping of \( R_{r-1} \) into \( S^{r-1} \) which sends each rotation \( r \) into the image under \( r \) of a fixed point \( y_0 \in S^{r-1} \). If \( p < 2r - 2 \) and \( f \) is a mapping of \( S^{r-1} \) into \( R_{r-1} \), then \( f \) defines a mapping \( F: S^{r-1} \times S^{r-1} \rightarrow S^{r-1} \) of type \( (πα, 1) \), where \( α \) is the element of \( π_{p-1}(R_{r-1}) \) represented by \( f \) and 1 is the element of \( π_{r-1}(S^{r-1}) \) represented by the identity map. Then \( F \) determines an element \( γ \) of \( π_{p+r-1}(S^r) \) as in the preceding paragraph, with \( H(γ) = \) the \( r \)-fold Einhängung \( Eπα \) of \( πα \). Since \( p < 2r - 2 \), \( E \) is an isomorphism\(^2\) and it follows that if \( πα \neq 0 \), then \( γ \neq 0 \). This result can be used to construct essential maps of \( S^n \) into \( S^r \) with \( n = 12, 14, 8k \) and \( 16k + 2 \), and \( r = 6, 7, 4k \) and \( 8k \), respectively.

If \( r = 2, 4, 8 \), Hurewicz and Steenrod\(^4\) have proved that \( π_n(S^r) \) is isomorphic with the direct sum \( π_n(S^{r-1}) + π_n(S^{r-1}) \). This isomorphism determines a homomorphism \( H' \) of \( π_n(S^r) \) into \( π_n(S^{2r-1}) \). It is then easy to see that if \( n < 3r - 3 \), then \( H' = H \).

3. Let \( X \) be an arcwise connected space, \( f \) a map of \( S^n \) into \( S^r \), and \( g \) a map of \( S^r \) into \( X \). The correspondence \((f, g) → gf\) defines an operation associating with \( α \in π_n(S^r) \), \( β \in π_n(X) \) an element \( β \cdot α \in π_n(X) \). It is known that the left distributive law \( β \cdot (α_1 + α_2) = β \cdot α_1 + β \cdot α_2 \) holds, while the corresponding right distributive law is in general false. Using the homomorphism \( H \) defined above, we can prove: if \( n < 3r - 3 \), then

\[
(β_1 + β_2) \cdot α = β_1 \cdot α + β_2 \cdot α + [β_1, β_2] \cdot H(α)
\]

where \([β_1, β_2]\) is the product defined by J. H. C. Whitehead.\(^3\)

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1. Consider the system of differential equations over the interval \( 0 \leq t \leq ∞ \)

\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, \ldots, n
\]  

(1)
and the perturbed system

\[ \frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} y_j + f_i(y_1, \ldots, y_n, t). \]  

(2)

We are interested in determining conditions on the \( f_i \) and \( a_{ij}(t) \) which will ensure that the solutions of (2) have properties similar to those of (1). The classical stability theory of Liapounoff, where (1) is regarded as an approximation to (2), for particular \( f_i \), is included in questions of this sort, as is also some fairly recent work of Cesari, treating the case where the \( f_i \) are linear in the \( y_i \). For the case of linear perturbation, a simple method of proof, sufficient also for the case of variable \( a_{ij} \), was obtained by the author, for the particular case of nth order linear differential equations. The same methods apply to linear systems.

Here, however, we are interested in a more general situation, and the purpose of this note is to indicate how Liapounoff's and Cesari's investigations are special cases of one more general result, which can be proved simply and directly by use of a classical method, the Picard iteration process.

For simplicity, we write the equation in vector form. Let \( x \) be the column vector with components \( x_i, i = 1, \ldots, n \), \( f(x, t) \) the column vector whose components are \( f_i(x_1, \ldots, x_n, t), i = 1, \ldots, n \), and let \( A \) be the matrix \( a_{ij}, i, j = 1, \ldots, n \).

Equation (1) becomes

\[ \frac{dx}{dt} = Ax \]  

(3)

and (2) becomes

\[ \frac{dy}{dt} = Ay + f(y, t). \]  

(4)

Furthermore, define

\[ \|x\| = (\sum_i |x_i|^2)^{1/2}. \]  

(5)

If \( \|x\| \) is bounded as \( t \to \infty \), we say that the solution is bounded. The following results can now be stated:

**Theorem 1**: If all the solutions of (3) are bounded, all the solutions of (4) are bounded, provided:

1. \( A \) is a constant matrix.
2. \( \|f(y, t)\| \leq c_m \phi(t) \) for \( \|y\| \leq m \), and
Theorem 2: The same result holds if (1) is replaced by (1):

\[ \int_0^t (\text{tr} A) \, dt \leq c < \infty, \quad 0 \leq t \leq \infty. \]

Conditions (2) and (3) are satisfied if \( f(y, t) \) has all components of the form \( g(t)h(y) \), where

\[ \int_0^\infty g(t) \, dt < \infty \]

and \( h(y) \) has a continuous derivative with respect to \( y \) for \( 0 \leq y < \infty \).

Condition (3) is added to ensure that all solutions of the perturbed system are bounded. If it is a question of exhibiting perturbed solutions which are close to the original solution, the following result may be applied:

Theorem 3: If all solutions of (3) are bounded, there exist bounded solutions of (4), provided:

1. \( A \) is a constant matrix.
2. \( f(y, t) \) is a continuous function of \( y \), and

\[ \|f(y, t)\| \leq c_m \phi(t), \quad \int_0^\infty \phi(t) \, dt < \infty. \]

Furthermore, \( y \) can be chosen to have the same value as \( x \) at a point, \( t_0 \), and for \( t \geq t_0(\varepsilon) \), \( \|y - x\| \leq \varepsilon \). As in Theorem 2, this result can be extended to variable \( A \).

If a stronger requirement is imposed upon the matrix \( A \), a more liberal perturbation is allowed. Thus, if we insist that the characteristic roots of

\[ |A - \lambda I| = 0 \]

all have negative real part, it is sufficient to assume merely that \( \phi(\delta) \) and \( \psi(t) \) are sufficiently small, depending upon \( A \). It is more than enough that they tend to zero as \( t \to \infty \). If we impose this restriction on \( A \), and the condition on the \( f_i \) that they are power series in the \( y_k \) with constant coefficients, beginning with quadratic terms, we have a fundamental theorem of Liapounoff. It does not seem to have been previously noticed that the hypothesis of Cesari's theorem (essentially (2) of Theorem 1) can be weakened to a boundedness condition if the above restriction is made upon \( A \).

We shall sketch the method of proof. The differential equation (4) is transformed into the integral equation

\[ y = x + \int_0^t X(t)X^{-1}(t_i)f(y(t_i), t_i)\, dt_i, \]

where \( X(t) \) is the matrix solution of (3) satisfying \( X(0) = I \).

If \( A \) is constant, the equation is even simpler, and becomes

\[ y = x + \int_0^t X(t - t_i)f(y(t_i), t_i)\, dt_i. \]

Conditions (2) and (3) are satisfied if \( f(y, t) \) has all components of the form \( g(t)h(y) \), where

\[ \int_0^\infty g(t) \, dt < \infty \]

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If \( A \) is constant, the equation is even simpler, and becomes

\[ y = x + \int_0^t X(t - t_i)f(y(t_i), t_i)\, dt_i. \]
The conditions imposed on \( f(y, t) \) and \( A \) are now sufficient to show convergence of the sequence

\[
y_0 = x \\
y_{n+1} = x + \int_0^t X(t)X^{-1}(t_1)f(y_n(t_1), t_1)dt_1
\]

using the classical methods.

Theorem 3 requires use of the Birkhoff-Kellogg\(^2\) fixed point theorem applied to (8).

The Liapounoff result can be proved using the character of the solution of (3), for \( A \) restricted as above, and the quadratic character of the \( f_i \).

2. Using the elementary methods mentioned previously, the following theorem can be obtained:

**Theorem 4:** All solutions of

\[
\frac{d^2x}{dt^2} = (A + B)x
\]

are bounded, provided

1. \( A \) is symmetric with negative characteristic roots.
2. \( \|B\| = \sum |b_{ij}|^{1/2} \leq b. \)
3. \( \int_0^\infty \left| \frac{dB}{dt} \right| dt < \infty. \)

The constant \( b \) depends upon \( A \).

**Corollary:** All solutions of

\[
y + (a^2 + \phi(x))y = 0
\]

are bounded, if

1. \( \int_0^\infty |f'(x)| dx < \infty. \)
2. \( |\phi(x)| \leq b < a^2. \)