probability 1. (Note that we do not assume any relation of dependence or independence between $X_n^{(r)}$ and $X_m^{(s)}$ for $m \neq n$.)


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**GREEN'S FUNCTIONS FOR LINEAR DIFFERENTIAL SYSTEMS OF INFINITE ORDER**

By D. V. Widder

In these PROCEEDINGS1 the author sketched a theory whereby a special differential equation of infinite order

$$\frac{\sin \pi D}{\pi} y(x) = \varphi(x) \quad (1)$$

could be solved by use of a Green's function. The operator on the left of this equation is interpreted to mean

$$\lim_{\pi \rightarrow \infty} D \left(1 - \frac{D^2_m}{m^2}\right) \ldots \left(1 - \frac{D^2_n}{n^2}\right) y(x),$$

where $D$ is the operation of differentiation with respect to $x$. In preparing the details of this theory the author discovered that much more general differential equations could be treated by the same method. It is the purpose of the present note to outline this more general theory.

We define an entire function $E(s)$ as follows:

$$E(s) = s \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right). \quad (2)$$

where the constants $a_k$ are real and such that

$$0 < a_1 < a_2 < \ldots. \quad (3)$$

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty. \quad (4)$$

Consider now the differential system

$$E(D)y(x) = \varphi(x) \quad (5)$$
We are assuming that \( \varphi(x) \) is a given continuous function which is absolutely integrable on \(( -\infty, \infty)\). In particular, if \( a_k = k \), equation (5) reduces to equation (1).

It is natural to define the Green's function, \( G(x) \), for the system (5) (6) (7) as the limit of the Green's function, \( G_{2n+1}(x) \), of the "truncated" system

\[
D \prod_{k=1}^{n} \left( 1 - \frac{D^2}{a_k^2} \right) y(x) = \varphi(x)
\]

with boundary conditions (6) and (7). We recall the definition of \( G_{2n+1}(x) \). It satisfies the semihomogeneous system

\[
D \prod_{k=1}^{n} \left( 1 - \frac{D^2}{a_k^2} \right) y(x) = 0
\]

\[
y(-\infty) = 0, \quad y(\infty) = 1
\]

for all \( x \) different from zero. It is continuous with its first \((2n-1)\) derivatives for all \( x \). Its \( 2n \)th derivative is continuous except at \( x = 0 \), where

\[
y^{(2n)}(0^+) - y^{(2n)}(0^-) = (-1)^n a_1^2 a_2^2 \cdots a_n^2.
\]

It is easy to see that these conditions determine \( G_{2n+1}(x) \) uniquely and that the unique solution of the system (8) (6) (7) is

\[
y(x) = \int_{-\infty}^{\infty} G_{2n+1}(x - t) \varphi(t) dt.
\]

We can, in fact, write down an explicit integral formula for \( G_{2n+1}(x) \):

\[
G_{2n+1}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} \frac{ds}{s} \prod_{k=1}^{n} \left( 1 - \frac{s^2}{a_k^2} \right)
\]

This integral can in turn be evaluated by use of the calculus of residues, and is two distinct linear combinations of exponential functions in the two intervals \((-\infty, 0)\) and \((0, \infty)\).

We are next able to show that \( G_{2n+1}(x) \) tends to a limit \( G(x) \) as \( n \) becomes infinite, and that this function yields a solution

\[
y(x) = \int_{-\infty}^{\infty} G(x - t) \varphi(t) dt
\]

of the system (5) (6) (7). In the course of the proof of these facts, we show that the reciprocal of the function \( E(s) \) is a bilateral Laplace transform.

\[
\frac{1}{E(s)} = \int_{-\infty}^{\infty} e^{-st} G(t) dt,
\]
convergent in the interval $0 < s < a_1$. In fact the determining function, as we have indicated by the notation, is the Green's function described above.

We summarize the above results in the following theorem.

**Theorem.** If $\varphi(x)eC\cdot L$ in $(-\infty < x < \infty)$, if $E(s)$ is an entire function defined by (2) (3) (4), and if

$$G(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{E(s)} ds$$  

$0 < c < a_1$,

then the function $y(x)$ defined by (9) satisfies the system (5) (6) (7).

We have thus obtained a real linear differential inversion formula for the general Faltung-equation (9). In particular, if $a_k = k$, $G(x)$ becomes

$$G(x) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{\sin \pi s} ds = \frac{1}{1 + e^{-x}}$$

and equation (9) becomes

$$y(x) = \int_{-\infty}^{\infty} \varphi(t) e^{-t} \frac{e^{-t}}{e^{-t} + e^{-x}} dt,$$

or

$$f(x) = y \left( \log \frac{1}{x} \right) = \int_{0}^{\infty} \varphi \left( \log \frac{1}{t} \right) \frac{1}{x + t} dt$$ (10)

after an exponential change of variable. The same exponential change of variable applied to the differential operator (1) shows that

$$\lim_{k \to \infty} \frac{(-1)^{k-1}}{k!(k - 2)!} \left[ x^{2k-1} f^{(k-1)}(x) \right] k = \varphi \left( \log \frac{1}{x} \right).$$

We thus rediscover, as a special case of the present theory, the author's real inversion formula for the Stieltjes transform. In the light of this general theory the original discovery of that formula seems somewhat fortuitous, depending as it did on the Laplace asymptotic evaluation of an integral of the form

$$\int_{0}^{\infty} [g(t)]^{n} \varphi(t) dt.$$ (11)

It was the special nature of the constants $a_k$ as integers that introduced powers of a function $g(t)$. We are now able to retain the essential features of the method even though the integral is now replaced by another which no longer involves powers of a function. One of the outstanding features of the present theory is that it enables one to set up a whole new class of inversion formulas for integral transforms. One may either discover the
kernel of the transform from a given inversion operator, or, conversely, set up the inversion operator when the kernel is given. The present approach is related to the fundamental work of N. Wiener on the operational calculus. It should be observed, however, that there is here no appeal to the $L^2$-Fourier transform theory.

2 See, for example, M. Bocher, *Leçons sur les méthodes de Sturm*, Paris (1917), Chap. V.

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**SPECIAL VALUES OF $e^{x^2}$, COSH ($k\pi$) AND SINH ($k\pi$) TO 136 FIGURES**

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The present investigation was undertaken with two primary objects in view, namely, to obtain certain heretofore uncomputed values of $e^{x^2}$, cosh ($k\pi$) and sinh ($k\pi$), and to extend appreciably the degree of approximation of data previously found by others and by the author.

The actual calculations were based on his 137-place table of logarithms because preliminary tests of available infinite series showed that their application would have necessitated a much greater expenditure of labor and time than the radix logarithms. For example, the 79th term of the series $e^x = \sum_{n=0}^{\infty} (x^n/n!)$, when $x = \pi/6$, has 139 zeros between the decimal point and the leading figures 70673. Seventy-nine terms would not have been deterrent but for $n > 2$ examination of page 84 of the invaluable *Index of Mathematical Tables* revealed the facts that the published values of $\pi^n$ required for the earlier terms of the series attained 41 figures in one dubious case and that for somewhat lower degrees of approximation only 32 consecutive entries were tabulated. Again, consider the series $e^{i\pi x - x}$ which was chosen by the author in an earlier article because $x$ has the rational value 1/2 when $\sin^{-1} x = \pi/6$. The general expression for the terms involving even powers of $x$ may be written

$$t_{2k+1} = (2^2 + 1)(4^2 + 1)(6^2 + 1)\ldots[(2k - 2)^2 + 1]x^{2k}/(2k!).$$

When $x = 1/2$ the course of this function of $k$ and the prohibitively slow convergence of the series may be seen from the following values of $t_{2k+1}$

- $t_{33} = 5.0977 \times 10^{-38}$
- $t_{163} = 1.5209 \times 10^{-42}$
- $t_{243} = 6.8957 \times 10^{-77}$
- $t_{323} =$