A UNIQUENESS THEOREM FOR EIGENFUNCTION EXPANSIONS

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Let \( w_n(x, y) \) be the complete normal orthogonal eigenfunctions of the boundary value problem

\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \mu w = 0 \tag{1}
\]

\[
w(x, y) = 0 \text{ on } C. \tag{2}
\]

Where \( C \) is the boundary of a regular region \( R \), the eigenvalues \( \mu_n \) being arranged in non-decreasing order of magnitude. If \( f(x, y) \) be an arbitrary summable function defined in \( R \), we may write

\[
f(x, y) \sim \sum a_n w_n(x, y) \tag{3}
\]

\[
a_n = \int_D \int f w_n \, dx \, dy \tag{4}
\]

the series on the right of (3) being called the *Fourier Eigenfunction Series* and \( a_n \) the *Fourier Coefficients* of \( f(x, y) \). I have studied elsewhere the problem of convergence and summability of a Fourier Eigenfunction Series. In this note I am interested in announcing a result on uniqueness of eigenfunction expansion. Actually, we have the following,

**Theorem.** Let us suppose we are given an eigenfunction series

\[
a_1 w_1(x, y) + a_2 w_2(x, y) + a_3 w_3(x, y) + \ldots, a_1, a_2, \ldots \text{ real} \tag{5}
\]

satisfying the following properties

(i) There exists a continuous function \( \phi(x, y) \) defined on \( D + C \) and vanishing on \( C \) such that

\[
\phi(x, y) \sim \sum \frac{a_n}{\mu_n} w_n(x, y) \tag{6}
\]

(ii) At every point in the region the series (5) is summable \( (J_1^2, \lambda_n)^2 \) to a bounded measurable function \( f(x, y) \), i.e.,

\[
\lim_{t \to 0} 4 \sum \frac{a_n}{\mu_n} \frac{J_1^2(\lambda_n t)}{(\lambda_n t)^2} = f(x, y), \quad \lambda_n = (\mu_n)^{1/r} \tag{7}
\]

the series on the left converging for \( t > 0 \).

Then (5) is the *Fourier eigenfunction series of* \( f(x, y) \).

We indicate the general outlines of the proof:
The sum of the convergent series on the left of (7) defines, for every $t > 0$, a linear transformation of $\phi$:

$$4 \sum a_n w_n \frac{J_1^2(\lambda_n t)}{(\lambda_n t)^2} = U_t(\phi) \quad \text{say} \quad (8)$$

Then we have the following lemmas:

**Lemma 1.** If for every $(x, y)$ in $D$, \( \lim_{t \to 0} U_t(\phi) \geq 0 \) then $\phi$ is subharmonic.

Similarly if \( \lim_{t \to 0} U_t(\phi) \leq 0 \) then $\phi$ is superharmonic.

So if $U_t(\phi) \to 0$ as $t \to 0$ then $\phi$ is harmonic.

**Lemma 2.** \( \lim U_t[\phi] \geq c \) for every $(x, y)$ in $R$ implies $U_t(\phi) \geq c$ for some region contained in $R$ with a similar conclusion if \( \lim U_t(\phi) \leq c \).

From these lemmas we deduce that if $|f(x, y)| \leq M$, hypotheses (i) and (ii) of the theorem imply

$$|U_t(\phi)| \leq M.$$  

Therefore

$$4a_n \frac{J_1^2(\lambda_n t)}{(\lambda_n t)^2} = \int \int R U_t(\phi) w dx dy$$

and letting $t \to 0$ we obtain the theorem.

We might add in conclusion, hypothesis (i) will be fulfilled if $a_n$ tends to zero as rapidly as $\mu_n^{-1}$ or $(\log \mu_n)^{-1}$ or $(\log \mu_n)^{-1}$ or $(\log \log \mu_n)^{-1}$, etc. Thus if $a_n$ satisfies any one of these conditions and $\sum a_n w_n(x, y)$ converges everywhere to zero then $a_n = 0$; $n = 1, 2, \ldots$

In the case of a double trigonometric series \( \sum_{\mu, \nu} C_{\mu, \nu} \exp i(\mu x + \nu y) \)

$C_{00} = 0$ the theorem takes the following form:

(i) \( \sum_{\mu, \nu} \frac{C_{\mu, \nu}}{\mu^2 + \nu^2} \exp i(\mu x + \nu y) \) is the F.S. of a continuous periodic function $\phi(x, y)$.

(ii) \( \lim_{t \to 0} \sum_{\mu, \nu} \frac{C_{\mu, \nu}}{\mu^2 + \nu^2} \frac{J_1^2(t \sqrt{\mu \nu})}{\mu^2 + \nu^2} \exp i(\mu x + \nu y) = f(x, y) \)

at every point $(x, y)$ where $f(x, y)$, is a bounded measurable function. Then the trigonometric series is the F.S. of $f(x, y)$.
