in the present note are intended only as illustrations of an extensive theory which the writer plans to develop elsewhere.

1 For detailed references to the literature, the reader may consult a forthcoming book: Marden, M., Geometry of the Zeros of a Polynomial, to be published as a volume of Mathematical Surveys by the American Mathematical Society.

2 Equation (8) is Cauchy’s integral formula for \( f'(z) \), involving an unusual term to include discontinuous \( u(x, y) \).

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SOME PROPERTIES OF ROTATIONAL FLOW OF A PERFECT GAS

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1. Introduction.—Complete analytic solutions of definite boundary value problems of compressible rotational flow are extremely difficult to obtain. Therefore, in the present stage of study of rotational gas flow the inverse, or semi-inverse, approach promises to prove fruitful. We propose to deduce and to study flow patterns satisfying the differential equations of gas flow and having throughout the field certain prescribed geometric, kinematic or physical properties; but will not make them obey any prescribed boundary conditions.

Usually entire systems of flow patterns, rather than individual solutions, result from this type of approach. If an imposed condition is proved not to be satisfied by any possible flow pattern, then this negative result is of significance, as it establishes a general property of all flows satisfying the equations of the problem.

In various fields of fluid mechanics the semi-inverse approach has been used with considerable success. It is sufficient to recall the contributions of Beltrami, Massotti, Jeffreys, Hamel, Oseen, Kampé de Fériet,\(^1\) Bateman and Tollmien.\(^2\) None of these investigations deals however with the rotational flow of a perfect gas.

The reason for this is probably that such an investigation—unless restricted to cases of uniform stagnation enthalpy—requires complicated and difficult eliminations if based on the familiar equations of the problem.

In the recent investigations of Munk and Prim\(^3\)\(^4\) this elimination is accomplished with complete generality as far as steady flow of a perfect gas is concerned. This makes the application of the semi-inverse approach to rotational gas flow practicable. Therefore the results of Munk and Prim are used throughout the present paper. The present paper is
a general preliminary report on our investigations, parts of which will be published in detail elsewhere.

2. Some Properties of the Plane Flow of a Perfect Gas.—A natural coordinate system for the analysis of problems of plane fluid motion employs as coordinate curves the streamlines (\( \xi = \text{const} \)) and their orthogonal trajectories (\( \eta = \text{const} \)). In this coordinate system the vorticity of the reduced velocity field is expressed by

\[
| \mathbf{curl} \mathbf{w} | = -\frac{1}{g_1 g_2} \frac{\partial}{\partial \xi} g_2 w
\]  

\((g_1 \, d\xi \text{ and } g_2 \, d\eta \text{ are components of the vector element of arc length}).

The vorticity equation and the continuity equation\(^4\) take on the following form:

\[
\frac{\partial}{\partial \eta} \left[ \frac{w^2}{1 - w^2} \frac{\partial \ln (g_2 w)}{\partial \xi} \right] = 0 \quad (2)
\]

\[
\frac{\partial}{\partial \eta} \left[ g_1 (1 - w^2)^\theta w \right] = 0. \quad (3)
\]

These equations enable us to settle certain general questions concerning plane gas flows.

We ask first under what conditions the velocity is constant along each streamline. Since the ultimate velocity is in every case constant along each streamline, our question is equivalent to asking under what conditions the reduced velocity is constant along each streamline.

Introduction of the requirement \( w = w(\xi) \) into equations (2) and (3) yields the geometric conditions:

\[
\frac{\partial^2 \ln g_2}{\partial \xi \partial \eta} = 0 \quad \text{or} \quad g_2 = a(\xi) b(\eta)
\]

\[
\frac{\partial \ln g_1}{\partial \eta} = 0 \quad \text{or} \quad g_1 = c(\xi).
\]

An analysis of this geometric limitation leads to the following theorem:

**Theorem 1.** In plane steady flow of a perfect gas, in absence of an external field of mass force, the velocity magnitude can have a constant value along each streamline only if the streamlines are concentric circles or parallel straight lines.

Obviously, in these cases in which \( w = \text{const} \) (and hence also \( v = \text{const} \)) along each streamline, \( \mathbf{curl} \mathbf{w} \) and \( \mathbf{curl} \mathbf{v} \) also have a constant magnitude along the same lines. The question arises whether the converse theorem is also true, that is, whether the only two dimensional flow patterns possible in a perfect gas such that the streamlines are lines of equal vorticity are those in which they are also lines of equal velocity magnitude.
To settle this question we consider first flow with constant reduced vorticity magnitude along each streamline. Crocco's pressure theorem as reformulated for a general distribution of stagnation enthalpy requires that the pressure remain constant along each streamline if the reduced vorticity does so; that is, $p = p(\xi)$. On the other hand, according to a familiar relation

$$\frac{p}{p_0(\xi)} = (1 - w^2)^{\theta + 1}$$

($p_0$ denotes the stagnation pressure, which remains constant along each streamline between shock fronts). Hence $|\text{curl } \bar{\omega}| = f(\xi)$ implies that $w$ must be also a function of $\xi$ alone.

A very similar argument shows that if the lines of equal velocity magnitude (isovels) and the lines of equal vorticity (isocurls) of the reduced velocity field coincide, then they coincide also with the streamlines.

Starting from the vector identity

$$\text{curl } \bar{\omega} = a(\xi) \text{ curl } \bar{\omega} + \text{ grad } a(\xi) \times \bar{\omega}$$

we have transformed the above statements relating to the reduced velocity field into analogous statements concerning the actual velocity field and proved the following theorems:

**Theorem 2.** In plane steady flow of a perfect gas, in absence of an external field of mass forces, constant vorticity along each streamline implies constant velocity along the same lines.

**Theorem 3.** In plane steady flow of a perfect gas, in absence of an external field of mass forces, the coincidence of the isovels and the isocurls implies that they coincide also with the streamlines.

Our three theorems prove that all flow fields having coincident isovels and streamlines, or coincident isocurls and streamlines, or coincident isovels and isocurls are restricted to flows the streamlines of which are concentric circles or parallel straight lines.

In a paper by R. Prim the question is investigated whether the streamline pattern can form, with its orthogonal trajectories, an isothermal net, that is a net in which $g_1 = g_2$. The following theorem is established.

**Theorem 4.** In plane steady flow of a perfect gas in absence of an external field of mass forces, the streamlines and their orthogonal trajectories can form an isothermal net only if the streamlines are concentric circles, parallel straight lines or radial straight lines: the last case being possible only if the reduced velocity field is irrotational.

It will be instructive to compare the geometric restrictions imposed by the above four conditions upon the pattern of gas flow with the corresponding restrictions imposed upon incompressible non-viscous and in-
compressible viscous flow. (For all three types of idealized fluids absence of an external field of mass forces is assumed.)

<table>
<thead>
<tr>
<th>NON-VISCOUS INCOMPRESSIBLE FLUID (&quot;IDEAL FLUID&quot;)</th>
<th>VISCOUS INCOMPRESSIBLE FLUID</th>
<th>PERFECT GAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Streamlines coincide with isovels</td>
<td>Streamlines concentric circles or parallel straight lines</td>
<td>Streamlines concentric circles or parallel straight lines</td>
</tr>
<tr>
<td>Isocurls coincide with streamlines</td>
<td>Any rotational plane flow has this property (Helmholtz)</td>
<td>Answer is not known</td>
</tr>
<tr>
<td>Isocurls coincide with isovels</td>
<td>Streamlines concentric circles or parallel straight lines</td>
<td>Streamlines concentric circles or parallel straight lines</td>
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<tr>
<td>Streamlines and their orthogonal trajectories form an isothermal net ((g_1 = g_2))</td>
<td>(1) Any irrotational flow only if streamlines concentric circles or parallel straight lines</td>
<td>(1) Irrotational flow: streamlines concentric circles or parallel straight lines</td>
</tr>
<tr>
<td></td>
<td>(2) Rotational flow only if streamlines log-spiral or their limiting cases (concentric circles, radial straight lines, parallel straight lines)</td>
<td>(2) Rotational flow: streamlines concentric circles or parallel straight lines</td>
</tr>
</tbody>
</table>

3. *On Flow Fields for Which, in a Cylindrical Isothermal Coördinate System All Three Velocity Components Depend Only on One Coördinate.—* For irrotational, plane gas flow this question was investigated by Tollmien. We have extended the study to rotational flow, at the same time allowing an additional velocity component parallel to the elements of the cylindric coördinate surfaces. In one respect, however, our investigation is less general than that of Tollmien: we have investigated only systems of such cylindrical coördinate surfaces which in the \(xy\) plane form an *isothermal* net, while Tollmien admitted orthogonal nets of any kind.

The main result of the investigation can be summarized in the theorem:

**Theorem 5.** *The only isothermal cylindrical system of coordinates for which the reduced velocity components of any steady non-parallel flow of a perfect gas, in absence of an external field of mass forces, are functions of one of the isothermal coordinates only are formed by logarithmic spirals and their limiting cases.* Thus the coördinate systems satisfying our kinematical
conditions are exactly the same as those found by Tollmien for plane potential flows. With reference to this spiral-cylindric coördinate system the differential equations can be reduced to a pair of ordinary non-linear differential equations of the first order.

4. Massotti Flow.—Massotti sought incompressible, non-viscous flow patterns for which \( v_z = 0 \), and for which at the same time \( (\text{curl } \vec{v})_z = 0 \).

There is a wide variety of compressible flow patterns the reduced velocity vector of which satisfies the corresponding relations \( w_z = 0 \) and \( (\text{curl } \vec{w})_z = 0 \). We satisfy the latter equation by setting

\[
\begin{align*}
\vec{w}_x &= \frac{\partial \phi}{\partial x}, \\
\vec{w}_y &= \frac{\partial \phi}{\partial y}.
\end{align*}
\]

Hence

\[
\vec{w}^2 = \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2
\]

and the continuity equation takes on the form:

\[
\Delta_{x,y} \phi [1 - \vec{w}^2] = \beta \left[ \frac{\partial \phi}{\partial x} \frac{\partial \vec{w}^2}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \vec{w}^2}{\partial y} \right].
\]

Our vorticity equation, \( \text{curl} \left[ \frac{\vec{w} \times \text{curl } \vec{w}}{1 - \vec{w}^2} \right] = 0 \), is automatically satisfied as \( \vec{w} \times \text{curl } \vec{w} \) is obviously perpendicular to the \( x, y \) plane and equal to \( \frac{1}{2} (\partial \vec{w}^2 / \partial z) \).

Hence the only condition imposed upon \( \phi \) is the continuity equation which is identical with that of irrotational plane compressible flow. We can immediately see that a wide variety of Massotti flows can be built from families of plane irrotational flows in the following way. Assume \( \phi(x, y, c_1, c_2, \ldots, c_n) \) is the potential of the reduced field of an \( n \)-parameter family of irrotational flows. Replace each of these parameters by an arbitrary function of \( z \); the \( \phi \) thus obtained can be considered as the potential from which the \( x \) and \( y \) components of families of reduced velocity fields of the Massotti type can be derived.

For example, we have based upon the simple irrotational Prandtl-Meyer flow a pseudo-plane rotational flow past a developable surface.

5. Beltrami Flows and Other Investigations.—Beltrami\(^7\) was the first to investigate velocity fields which satisfy the condition \( \vec{v} \times \text{curl } \vec{v} = 0 \). In recent memoranda\(^8, 9, 10\) the present writers investigate flows of perfect gases which satisfy this relation (Beltrami Flow Proper). They found it fruitful to study also the broader family of flows for which \( \vec{w} \times \text{curl } \vec{w} = 0 \) (Generalized Beltrami Flow).

Further semi-inverse investigations under way include the study of gas
flow patterns for which all streamlines are straight lines or for which the hodograph space degenerates into a surface or a curve. In addition, the basis of these semi-inverse methods, the equations of Munk and Prim, are being generalized to include certain classes of non-steady flow and of compressible substances with a more general state equation.

1 See, for example, Berker, A. Ratib, "Sur quelques cas d'intégration des équation du mouvement d'un fluide visqueux incompressible," Paris et Lille, 1936.


REPRODUCTIVE DIAPAUSE IN DROSOPHILA ROBUSTA

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Introduction.—Studies of microevolutionary processes in natural populations of Drosophila have been hampered by a lack of ecological data. For most species of the genus, the basic facts concerning population size, migration, longevity, specific breeding sites, length of breeding season and nature of overwintering populations are unknown. Workers in this field have not ignored these subjects1, 2, 3, but it is clear that preoccupation with other basic problems has generally held up experiments specifically planned to answer these questions. It is, nevertheless, true that a full understanding of natural selection, genetic drift and other processes, as they are oper-