terminating graphs $r^{(n)}$ which contain at least one particle in generation $n$, but none in the next.

By an interval of order $n$ is meant the set $\gamma(\tau^{(n)})$ of all graphs $\gamma$ such that $\gamma_n = \tau^{(n)}_n$ where $\tau^{(n)}$ is a particular graph in $T_n$. The only interval of order 0 is $\Gamma_1$ itself. We now define interval to be either $\theta$, or any single graph $\gamma$, terminating or not, or any interval of order $n$, $n = 0, 1, \ldots$ It is easy to see that axioms I.1, 2, 3 are satisfied. Indeed we have the simple property that the intersection $ij$ of two intervals is either $\theta$ or $i$ or $j$.

4. Measure in the Space of Graphs.—While the notions of graph, distance and interval are geometric in character and depend only on the number $t$ of types, measure may be introduced in various ways. One of the simplest, but by no means the only one to which the theory has been applied, may be imposed as follows. Suppose we assign a probability $p(i; j_1, \ldots, j_l)$ to the event that a particle of type $i$ should produce, upon transformation, $j_1 + \ldots + j_l$ particles, $j$, of type $v$. Then every segment $\gamma_n$ of a graph $\gamma$ has an associated probability $p(\gamma_n)$ that the event described by $\gamma_n$ should occur. If we assign to intervals a measure by

$$m(\tau^{(n)}) = p(\tau^{(n)}_n), m(\gamma(\tau^{(n)})) = p(\tau^{(n)}_n),$$

and

$$m(\gamma) = \lim_{\gamma \to \gamma_n} p(\gamma_n),$$

for $\gamma$ non-terminating it turns out that M.1, 2, 3 are satisfied and thus Borel sets based on our intervals are measurable in the sense of §2. (Non-compactness of intervals makes M.3 non-trivial.)

The procedure applies in much more general systems where the transition probabilities are functions of the time.

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ON THE ZEROS OF THE DERIVATIVE OF AN ENTIRE FUNCTION OF FINITE GENRE

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Let $E(z)$ denote an entire function of genre $p$:

$$E(z) = e^{\sum_{l=1}^{\infty} e^{2\pi i} \Pi_{j=1}^{l-1} [1 - (z/z_j)] \exp \sum_{k=1}^{l-1} [(z/z_j)^k/k],}
$$

where $P(z)$ is a polynomial of degree $p_1$ and $p = \max (p_1, g)$. Clearly, the zeros of its derivative $E'(z)$ depend not only upon the zeros $z_j$ of $E(z)$ but
also upon the polynomial \( P(z) \). This contrasts with the fact that the zeros of the derivative of a polynomial depend only upon the zeros of the polynomial. It accounts for the difficulty experienced by some previous investigators who, seeking to extend to entire functions the well-known results on the zeros of the derivative of a polynomial, found it necessary to assume \( \Re P(z) = 0 \) or to restrict \( \rho \) to the values 0 or 1.\(^1\)

In the present note, the difficulty due to the \( \Re P(z) \) is circumvented by establishing for \( E'(z) \) the following new representation.

**Theorem 1.** Let \( E(z) \) be an entire function of genre \( \rho \). Let \( z_0 = 0, \ z_1, z_2, \ldots \) denote the zeros of \( E(z) \) with the multiplicities \( m_0, m_1, m_2, \ldots, \) respectively, and let \( Z_1, Z_2, \ldots, Z_p \) denote any \( p \) zeros of \( E'(z) \). Then

\[
E'(z) = E(z)\sum_{j=0}^{p-1} [m_j/(z - z_j)]\prod_{k=1}^{p}[(Z_k - z)/(Z_k - z_j)].
\]

Theorem 1 may be proved by extending to meromorphic functions our recent results\(^2\) on the zeros of rational functions of the form

\[
F(z) = \sum_{j=0}^{p} A_j z^j + \sum_{j=1}^{p} m_j/(z - z_j),
\]

where the \( A_j \) are arbitrary complex constants and the \( m_j \) are points in a convex sector. It may also be proved directly by eliminating \( P(z) \) from \( E'(z) \) by use of the equations \( E'(Z_k) = 0, \ k = 1, 2, \ldots, p \).

An immediate consequence of Theorem 1 is

**Theorem 2.** If \( Z_1, Z_2, \ldots, Z_{p+1} \) are any \( p + 1 \) zeros of the function \( E'(z) \), then

\[
\sum_{j=0}^{p} m_j/[(Z_1 - z_j)(Z_2 - z_j)\ldots(Z_{p+1} - z_j)] = 0. \quad (1)
\]

The study of the argument of each term in equation (1) leads now to

**Theorem 3.** Let \( K \) denote the smallest convex infinite region which encloses all the zeros of \( E(z) \) and let \( S(K, \psi) \) denote the star-shaped region comprised of all points from which \( K \) subtends an angle of at least \( \psi = \pi/(\rho + 1) \). Then at most \( \rho \) zeros of \( E'(z) \) lie outside of \( S(K, \psi) \).

When \( \rho = 0 \), the region \( S(K, \psi) \) coincides with \( K \). Thus Theorem 3 is seen to be a generalization of the Lucas Theorem that any convex region enclosing all the zeros of a polynomial also encloses all the zeros of its derivative.

In particular, if \( K \) is chosen as a convex sector, the following result is obtained.

**Theorem 4.** Let \( \alpha \) and \( \beta \) be non-negative numbers such that \( \alpha \leq \beta, \ \alpha + \beta = \pi/(\rho + 1) \). If all the zeros of \( E(z) \) lie in the sector \( | \arg z | \leq \alpha \), then at most \( \rho \) zeros of \( E'(z) \) lie in the sector \( | \arg (-z) | \leq \beta \).

Equation (1) also leads to an elementary proof of the following result of Laguerre and Borel.\(^3\)

**Theorem 5.** Let \( E(z) \) be a real entire function with only real zeros. Then
$E'(z)/E(z)$ has at most $p$ real and non-real zeros in excess of a single zero between each pair of successive zeros of $E(z)$.

All of the above theorems remain valid if $E'(z)$ is replaced by $H(z) = E'(z) + Q(z)E(z)$ where $Q(z)$ is an arbitrary (real, in the case of Theorem 5) polynomial of degree not exceeding $p - 1$. For, the function $E_1(z) = E(z) \exp \int Q(z)dz$ is an entire function of genre $p$ possessing the same zeros as $E(z)$ and its derivative $E_1'(z) = H(z) \exp \int Q(z)dz$.

A more detailed account of the above results, together with additional applications, will be published elsewhere at a later date.


3 E. Borel, Lecons sur les fonctions entières, Paris, 1921, pp. 36-44.

TWENTY EXACT FACTORIALS BETWEEN 304! AND 401!

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In the year 1944 the author published privately a little book entitled Exact Values of the First 200 Factorials. Subsequently he computed with great care the exact values of $n!$ from $n = 201$ to $n = 300$. The data of this third century have not appeared in print. One consultable copy has been deposited in the library of Brown University, Providence 12, Rhode Island, and another copy is in the possession of Doctor J. C. P. Miller, Technical Director of Scientific Computing Service Limited, 23 Bedford Square, London, W.C. 1, England.

Recently the author has computed a skeleton table of 42 exact factorials beginning with 303! and ending with 400!. This table was built up by first calculating the values of $n!$ for which $n + 1$ was one of the 17 primes from $n = 307$ to $n = 401$, so that Wilson’s theorem could be applied as a more exacting check in addition to congruence testing with moduli such as $10^8 + 1$, $10^8 + 1$, etc. Incidentally the values of 350!, 372!, 375!, 378! and 400! as found by the author in February, 1945, were reproduced identically in the work performed three years later. In order to make a few of these arithmetical constants available to other investigators requiring exact values in the fourth century of $n!$ the following table of equally spaced but non-consecutive data is presented.