gard to the fermentation of alpha methyl glucoside, maltose and sucrose. Regularly segregating SU/su alleles can be introduced with the stock and members of each of the above four classes incapable of fermenting sucrose are available.

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Lindegren, Carl C., and Gertrude Lindegren (in press).

Dr. Caroline Raut has eliminated the possibility that the MA and MG genes control permeability to maltose and alpha methyl glucoside by showing that the activities of enzyme preparations correspond to the fermentation diagnoses. These facts have led to the conclusion that the MA gene controls the rate of hydrolysis of maltose and has little or no effect on hydrolysis of alpha methyl glucoside and vice versa.

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**A THEOREM ON CONVEX BODIES OF THE BRUNN-MINKOWSKI TYPE**

**BY HERBERT BUSEMANN**

**UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES**

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The present note establishes a theorem on convex bodies in the n-dimensional Euclidean space $E^n$ which is a counterpart to the Brunn-Minkowski Theorem. The following formulation of the latter [compare Bonnesen-Fenchel, *Theorie der konvexen Körper*, Berlin, 1934 (quoted as B. F.), pp. 71, 72] will exhibit the analogy:

In $E^n$ let $K$ be a convex body with interior points and $P$ a two-dimensional half-plane bounded by the line $L$. Let the hyperplane normal to $L$ intersect $K$ in the non-empty set $H \cap K$ with $(n-1)$-dimensional volume $V(H \cap K)$. If $V^{1/(n-1)} (H \cap K)$ is laid off from the point $H \cap L$ on the ray $H \cap P$, then the resulting curve in $P$ is convex (and turns its concavity towards $L$).

The present theorem is concerned with the intersection of $K$ by pencils of non-parallel hyperplanes:
1. In $E^n$ let $K$ be a convex body with interior points, $L$ an $(n - 2)$-dimensional linear space which intersects $K$, and $P$ a 2-dimensional space normal to $L$ at a point $0$. Any half-hyperplane $H$ bounded by $L$ intersects $K$ in a non-empty set $H \cap K$. If the $(n - 1)$-dimensional volume $V(H \cap K)$ of $H \cap K$ is laid off from $0$ on the ray $H \cap P$ then the resulting curve $C$ is convex (C contains 0 in its interior if $L$ contains interior points of $K$, otherwise $P$ lies on $C$).

Both theorems remain true, but are trivial if $K$ has no interior points.

Notice the following Corollary of I:

II. Let $K$ be a convex body in $E^n$ with 0 as interior point and center. If for any hyperplane $H$ through 0 the volume $V(H \cap K)$ is laid off from 0 on the normal to $H$ at 0 (in both directions), then the resulting surface $S$ is convex.

II follows from I because it suffices to know that the intersection of $S$ with a two-dimensional plane $P$ through 0 is a convex curve $C'$. But $C'$ originates from the $C$ of I by revolving $C$ in $P$ about 0 through $\pi/2$ and then dilating it in the ratio 2:1.

The result II is decisive for the theory of area in Finsler spaces and more specifically in Minkowski (or finite dimensional Banach) spaces. Its applications will be found in the author's series of articles on "Curvature and Angle in Non-Riemannian Spaces" which is appearing elsewhere.

Theorem I is trivial for $n = 2$, so that $n \geq 3$ will be assumed in the proof. It may also be assumed that $L$ contains interior points of $K$, because the other case follows from this one by an obvious limit process. Then $V(H \cap K) > 0$ for every half-hyperplane $H$ through $L$.

Let $r$ and $\bar{r}$ be non-opposite rays in $P$ with origin 0, and $q$ a ray with origin 0 in the angle with legs $r$ and $\bar{r}$. Let the segment connecting the points $r \cap s \in (r, s \neq 0)$ intersect $q$ at $q$. If $\angle (r, q) = \alpha'$, $\angle (q, \bar{r}) = \beta'$ then the fact that the area of the triangle $ors$ equals the sum of the areas of the triangles $orq$ and $oqs$ yields

$$or \cdot os \sin(\alpha' + \beta') = or \cdot oq \sin \alpha' + oq \cdot os \sin \beta',$$

hence

$$oq = or \cdot os \sin(\alpha' + \beta')/(or \sin \alpha' + os \sin \beta'). \quad (1)$$

Let $H_r, H_s, H_q$ denote the half-hyperplanes bounded by $L$ and through $r, \bar{r}, q$, respectively. Then (1) shows that I is equivalent to the statement

$$V(H_r \cap K) \geq \frac{\sin(\alpha' + \beta')V(H_r \cap K)\sin(\alpha' + \beta')}{\sin(\alpha' \sin(\alpha' + \beta') + \sin \beta'V(H_s \cap K)} \quad (2)$$

With the notations $\alpha = \sin \alpha'$, $\beta = \sin \beta'$, $\delta = \sin(\alpha' + \beta')$ and $V(H_r \cap K) = D$, $V(H_s \cap K) = A$, $V(H_q \cap K) = B$, the relation (2) becomes
\[ D \geq \delta AB/(\alpha A + \beta B). \] (3)

Let \( \overline{A_x(B_y)} \) denote the set at which the \((n - 2)\)-dimensional space parallel to \( L \) through the point \( r \) with \( or = x \) on \( r \) \((s \text{ with } os = y \text{ on } f)\) intersects \( K \). If \( A_x(B_y) \) is the \((n - 2)\)-dimensional measure of \( \overline{A_x(B_y)} \), then

\[ A = \int A_x \, dx \quad B = \int B_y \, dy. \]

Following the original idea of Brunn for the proof of the Brunn-Minkowski Theorem, we map the interval of \( r \) for which \( A_x \neq 0 \) on the interval of \( f \) for which \( B_y \neq 0 \) through the relations

\[ \rho A = \int_0^\rho A_x \, dt \quad \rho B = \int_0^\rho B_y \, dt, \quad 0 \leq \rho \leq 1 \] (4)

so that

\[ \frac{dx}{d\rho} = \frac{A}{A_x} \quad \frac{dy}{d\rho} = \frac{B}{B_y}, \quad 0 \leq \rho < 1. \] (5)

Since \( K \) is convex it contains for every \( \rho \) the convex closure \( C_\rho \) of \( \overline{A_x(\rho)} \) and \( \overline{B_y(\rho)} \). Since \( A_x \) and \( B_y \) lie in \((n - 2)\)-dimensional spaces parallel to \( L \), the set \( C_\rho \) lies in the hyperplane \( H_\rho \) through these two spaces. \( H_\rho \) intersects \( H_q \) in an \((n - 2)\)-dimensional space parallel to \( L \) and \( q \) itself in a point whose distance \( u(\rho) \) from 0 is by (1)

\[ u(\rho) = \delta x(\rho)y(\rho)/[ax(\rho) + \beta y(\rho)], \] (6)

hence by (5)

\[ \frac{du}{d\rho} = \delta \frac{ax^2y' + \beta y^2x'}{(ax + \beta y)^2} = \delta \frac{\alpha Bx^2B_y^{-1} + \beta Ay^2A_x^{-1}}{(ax + \beta y)^2}. \] (7)

The intersection \( \overline{C_{u(\rho)}} \) of \( C_\rho \) with \( H_q \) lies in \( \overline{D} = H_q \cap K \) because \( \overline{C_{u(\rho)}} = C_\rho \cap H_q \subset K \cap H_q \cap H_\rho = \overline{D} \cap H_\rho \). Therefore, if \( C_{u(\rho)} \) is the \((n - 2)\)-dimensional measure of \( \overline{C_{u(\rho)}} \),

\[ D \geq \int C_u \, du = \delta \int_0^1 C_{u(\rho)} \frac{\alpha Bx^2B_y^{-1} + \beta Ay^2A_x^{-1}}{(ax + \beta y)^2} \, d\rho. \] (8)

In the original triangle \( orq \) we find, if \( V = \angle ogr \), that

\[ or : rq = \sin V : \alpha, \quad os : qs = \sin (\pi - V) : \beta, \]

hence

\[ rq : qs = or : \alpha : os : \beta. \] (9)

If (9) is applied to the points \( q, r, s \) with \( oq = u(\rho), \ or = x(\rho), \ os = y(\rho) \) then \( rq : qs \) equals the ratio of the distances \( d_r, d_s \) of \( H_\rho \cap H_q \cap H_\rho \cap H_q \cap H, \) and \( H_\rho \cap H_q \cap H_\rho \cap H_q \cap H. \)
Therefore the Brunn-Minkowski Theorem (compare B. F., p. 88) yields applied to $A_{x(p)}$, $B_{y(p)}$ and $C_{u(p)}$ that

$$C_{u}^{1/(n-2)} \geq \frac{\beta y}{\alpha x + \beta y} A_{x}^{1/(n-2)} + \frac{\alpha x}{\alpha x + \beta y} B_{y}^{1/(n-2)}$$

hence by (8) and (11)

$$D \geq \delta \int_{0}^{1} \left[ \beta y A_{x}^{1/(n-2)} + \alpha x B_{y}^{1/(n-2)} \right]^{n-2} (\alpha B x^{2} B_{y}^{-1} + \beta A y^{2} A_{x}^{-1}) (\alpha x + \beta y)^{-n} dp. \quad (12)$$

In order to prove (3) it suffices to show that the integrand in (12) is for $0 < \rho < 1$ not less than $A B (\alpha A + \beta B)^{-1}$. A similar procedure as in the proof of the Brunn-Minkowski Theorem in B. F., p. 89, will be used. If $\epsilon = (n - 1)^{-1}$, then

$$\beta y A_{x}^{1/(n-2)} + \alpha x B_{y}^{1/(n-2)} = y^{1+\epsilon} \beta A^{e} \left( \frac{A_{x}}{A^{1-\epsilon} y^{1-\epsilon}} \right)^{1/(n-2)} + x^{1+\epsilon} \alpha B^{e} \left( \frac{B_{y}}{B^{1-\epsilon} x^{1-\epsilon}} \right)^{1/(n-2)}$$

$$\frac{\alpha B x^{2}}{B_{y}} + \frac{\beta A y^{2}}{A x} = y^{1+\epsilon} \beta A^{e} \frac{A_{x}^{1-\epsilon} y^{1-\epsilon}}{A_{x}} + x^{1+\epsilon} \alpha B^{e} \frac{B^{1-\epsilon} x^{1-\epsilon}}{B_{y}}$$

Putting $t = x^{1+\epsilon} \alpha B^{e} [x^{1+\epsilon} \alpha B^{e} + y^{1+\epsilon} \beta A^{e}]^{-1}$ the integrand in (12) may be written as

$$\left[ (1 - t) \left( \frac{A_{x}}{A^{1-\epsilon} y^{1-\epsilon}} \right)^{1/(n-2)} + t \left( \frac{B_{y}}{B^{1-\epsilon} x^{1-\epsilon}} \right)^{1/(n-2)} \right]^{n-2} \cdot \left[ (1 - t) \frac{A_{x}^{1-\epsilon} y^{1-\epsilon}}{A_{x}} + t \frac{B^{1-\epsilon} x^{1-\epsilon}}{B_{y}} \right] \cdot \frac{(x^{1+\epsilon} \alpha B^{e} + y^{1+\epsilon} \beta A^{e})^{n-1}}{(\alpha x + \beta y)^{n}}. \quad (13)$$

By Jensen's Inequality (compare Hardy, Littlewood, Polya, Inequalities, Cambridge, 1934, Theorem 16, p. 26)

$$[(1 - t)a_{1}^{1/\nu} + ta_{2}^{1/\nu}] \geq [(1 - t)a_{1}^{-1} + ta_{2}^{-1}]^{-1}$$

for $0 < t < 1$, $a_{i} > 0$, $\nu > 0$ \quad (14)

and the equality sign holds only when $a_{1} = a_{2}$.

Application of (14) to (13) with $\nu = n - 2$ yields that the integrand in (12) is at least

$$\frac{(x^{1+\epsilon} \alpha B^{e} + y^{1+\epsilon} \beta A^{e})^{n-1}}{(\alpha x + \beta y)^{n}} = \left( \frac{\alpha x \cdot x^{e} B^{e} + \beta y \cdot y^{e} A^{e}}{\alpha x + \beta y} \right)^{n-1} \frac{1}{\alpha x + \beta y}. \quad (15)$$

Applying (14) again with $\nu = n - 1 = \epsilon^{-1}$ and $t = \beta y (\alpha x + \beta y)^{-1}$ to the
first factor on the right in (15) yields that (15), and therefore the integrand in (12), is at least

\[
\left(\frac{\alpha x^{-1}B^{-1} + \beta y^{-1}A^{-1}}{\alpha x + \beta y}\right)^{-1} \cdot \frac{1}{\alpha x + \beta y} = \frac{AB}{\alpha A + \beta B}
\]

which proves Theorem 1.

Conditions for the equality sign in (2) are important for several applications. Because of the continuity of the functions involved, the equality sign in (2) requires the equality sign in (11) and in the two cases of Jensen's Inequality. Moreover \( \mathfrak{C}_{w(\rho)} \) must coincide with \( D \cap H' \).

The equality sign holds in (11) only if the sets \( \mathfrak{A}_{x(\rho)} \) and \( \mathfrak{B}_{y(\rho)} \) are homothetic (see B. F., pp. 72 and 88). The condition \( a_1 = a_2 \) for equality in (14) yields in the two cases

\[
\frac{A_x}{A^{1-\varepsilon}}x^{1-\varepsilon} = \frac{B_y}{B^{1-\varepsilon}}y^{1-\varepsilon} \text{ and } xB = yA
\]

so that \( A_x = B_y \).

Therefore \( A_x \) and \( xB_y \) are not only homothetic but congruent. The relation \( x:A = y:B \) shows then that \( \mathfrak{A} = H, \cap K \) can be transformed into \( \mathfrak{B} = H, \cap K \) by revolving \( \mathfrak{A} \) about \( L \) through \( \alpha' + \beta' \) and dilating it at the ratio \( B:A \) in the direction of \( f \). If \( w' \) is the point of \( \mathfrak{B} \) into which the point \( w \) of \( \mathfrak{A} \) is mapped under this transformation, it is easily seen that the segments connecting \( w \) to \( w' \) form a convex set \( E \) when \( w \) traverses \( \mathfrak{A} \), so that \( E \) is the convex closure of \( \mathfrak{A} \cup \mathfrak{B} \). Clearly the equality sign holds in (2) for \( K = E \). Thus we find

III. The equality sign holds in (2) if and only if \( H, \cap K \) can be transformed into \( H, \cap K \) by a rotation about \( L \) and a subsequent dilatation in the direction of \( f \), and \( H, \cap K \) is the intersection of \( H, \cap K \) with the convex closure of \( H, \cap K \) and \( H, \cap K \).

From III and the fact that \( E \) is the union of the segments connecting \( w \) and \( w' \) we obtain the following addition to II:

IIa. The surface \( S \) in II is strictly convex if \( K \) is strictly convex.

But III shows that the strict convexity of \( K \) is not a necessary condition.

THE \( n \)-ALITY THEORY OF RINGS

BY ALFRED L. FOSTER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY

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1. Introduction.—The classic duality-symmetry which is exhibited by Boolean rings (and Boolean algebras), far from being characteristic of