The theory of integration presented in the earlier notes\(^1\) postulates an elementary integral in terms of which all other concepts are defined. We must therefore undertake to compare the general integrals and other mathematical objects which are associated with different elementary integrals. In this note we offer such a comparison, giving the appropriate general forms of such classical results as the Lebesgue decomposition theorem and the Radon-Nikodym theorem. This is also a suitable context in which to consider an unpublished definition of a general integral recently introduced by N. Bourbaki.\(^2\) As we shall see below, the general integral of Bourbaki differs in few of its essential properties from the one treated in these notes. By comparing the two integrals we obtain solutions to a number of problems raised by Theorem 18 of our second note and mentioned in the corresponding footnote.

On a fixed abstract set \(X\), let \(\mathcal{C}'\) and \(\mathcal{C}^*\) be families of elementary functions; and let \(E'\) and \(E^*\) be elementary integrations defined over the respective families. It is easy to verify that the contractions of \(E'\) and \(E^*\) to the family \(\mathcal{C} = \mathcal{C}' \cap \mathcal{C}^*\) share with them the postulated properties set out in I (1), I (2). Consequently the study of the relations between \(E'\) and \(E^*\) breaks naturally into two parts—the first consisting in the study of two elementary integrals defined over the same family of elementary functions (viz., the contractions of \(E'\) and \(E^*\) to \(\mathcal{C}\)); the second in the study of two elementary integrals, one of which is an extension or contraction of the other (viz., \(E'\) or \(E^*\) and its contraction to \(\mathcal{C}\)). We shall discuss these situations separately below.

As linear functionals the elementary integrals over a fixed family \(\mathcal{C}\) of elementary functions constitute an ordered additive semigroup admitting multiplication by positive real scalars: the definitions of \(\alpha E, E' + E^*\), and the ordering relation \(\leq\) are given by requiring that \((\alpha E)(f) = \alpha (E(f)), (E' + E^*)(f) = E'(f) + E^*(f), E'(f) \leq E^*(f)\) for all \(f\) in \(\mathcal{C}\); and the various needed properties, including those demanded in I (1) and I (2), can be verified without difficulty. We note in particular that if \(E = E' + E^*\) then \(E' \leq E\) and \(E^* \leq E\) and that if \(E' \leq E\) then there is a unique \(E^*\), given by \(E^*(f) = E(f) - E'(f)\), such that \(E = E' + E^*\). In order to examine the relations between given elementary integrals \(E'\) and \(E^*\), we shall first consider how each is related to the sum \(E = E' + E^*\) and then combine such information as we obtain in this way to yield results on the main problem. The various quantities and other mathematical objects associated with the
elementary integrals $E$, $E'$ and $E^*$ will be distinguished by the use of the corresponding primes. Most of our results are of such a nature that we shall confine attention to the case where $1 \in \mathcal{M}$, that is, where the constant function everywhere equal to 1 is measurable in the theory associated with the elementary integral $E$. For practical purposes, indeed, there would be no objection to assuming that the property I (3), which implies $1 \in \mathcal{M}$, holds for $\mathcal{E}$. All our deeper results depend upon the additional assumption that

(1) $\mathcal{E}$ contains a sequence $\{f_n\}$ such that $\{x; \sup_n |f_n(x)| = 0\}$ is a null set.$^3$

An example given by Saks$^4$ shows that this assumption, or a similar one, is indispensable. Now we may note without formal proof that

(2) the relation $E' \leq E$ on $\mathcal{E}$ implies $N' \leq N$, $\mathcal{E}' \supset \mathcal{E}$, $\mathcal{E}' \supset \mathcal{M}$, and $\mathcal{M}' \supset \mathcal{M}$; it also implies that $L' \leq L$ in $\mathcal{L}$.

These preliminaries settled, we proceed to discuss an important portion of the Radon-Nikodym theorem,$^4$ modifying for our purposes a method of proof originally introduced by J. v. Neumann.$^5$

(3) (Radon-Nikodym) if $E' \leq E$, there exists an essentially unique function $\phi'$ in $\mathcal{M}$, $0 \leq \phi' \leq 1$, such that $f \in \mathcal{L}$ if and only if $\phi'f \in \mathcal{E}$, while $L'(f) = L(\phi'f)$; in particular, $E'(f) = L(\phi'f)$ when $f \in \mathcal{E}$.

For $h$ and $k$ in $\mathcal{E}$ the product $hk$ is in $\mathcal{E}$. Hence $L'(hk)$ is a bilinear functional defined for all $h$ and $k$ in $\mathcal{E}$. Since $|L'(hk)| \leq L'(|hk|) \leq L(|hk|) \leq L(h^*k^{1/2}L(k)^{1/2}$, a standard theorem$^6$ about generalized Hilbert spaces asserts the existence of a continuous linear transformation $A$ of $\mathcal{E}$ into itself such that $L'(hk) = L((Ah)k)$. If $g$ is any bounded member of $\mathcal{M}$, then $gh$ and $kg$ are in $\mathcal{E}$. The equations $L((A(gh))k) = L'(((gh)k) = L'(g(k)) = L((Ah)gk) = L((g(Ah))k)$ imply that $A(gh) = gAh$ almost everywhere. In order to exploit the latter relation, we use (1) to construct a bounded function $g_0$ in $\mathcal{E}$ such that $\{x; g_0(x) = 0\}$ is a null set. Supposing (as we obviously may) that the sequence $\{f_n\}$ of (1) contains no null function, we put $f_0 = \sum_{n=1}^{\infty} 2^{-n} |f_n|/L(|f_n|) \in \mathcal{E}$ in harmony with I (12) and observe that $\{x; f_0(x) = 0\}$ is a null set. We can then put $g_0 = (\min (1, f_0))^{1/2}$. For any $h$ in $\mathcal{E}$ the function $h_n = \text{mid} (h, n, -n)$ is bounded and converges in $\mathcal{E}$ to $h$ as $n \to \infty$. Thus $g_0(Ah_n) = A(g_0h_n) = A(h_ng_0) = h_n(Ag_0) = (Ag_0)h_n$ and passage to the limit in the extreme terms yields $g_0(Ah) = (Ag_0)h$ almost everywhere. A measurable function $\phi'$ is defined by putting $\phi' = Ag_0/g_0$ almost everywhere. Evidently $Ah = \phi'h$. Hence if $f \in \mathcal{E}$ and $h = |f|^{1/2} \in \mathcal{E}$, $k = |f|^{1/2} \text{sgn} f \in \mathcal{E}$, we have $L'(f) = L'(hk) = L(\phi'hk) = L(\phi'f)$. The inequalities $0 \leq L'(f) \leq L(f)$ hold whenever $f \geq 0$ and imply with the help of (1) that $0 \leq \phi' \leq 1$ almost everywhere. Similarly,
if \( \phi' \) and \( \phi'' \) are two measurable functions such that \( L(\phi'_1f) = L'{}(f) = L(\phi''_1f) \) for all \( f \in \mathfrak{F} \), we have \( L((\phi'_1 - \phi''_2)f) = 0 \) for all such \( f \) and hence \( \phi'_1 = \phi''_2 \) almost everywhere. Thus \( \phi' \) is uniquely determined (in the obvious sense). In proving the remainder of the theorem, we make use of the definition of \( \mathfrak{F}' \). A function \( f \) is in \( \mathfrak{F}' \) if and only if to an arbitrary \( \epsilon > 0 \) there correspond functions \( g \) and \( f_n \) in \( \mathfrak{F} \) such that \( |f - g| \leq \sum \limits_{n=1}^{\infty} |f_n|, \sum \limits_{n=1}^{\infty} E'(|f_n|) \leq \epsilon \).

Clearly we have \( |L'(f) - L'(g)| \leq \epsilon \) under these circumstances. Since

\[
|\phi'f - \phi'g| \leq \sum \limits_{n=1}^{\infty} |\phi'|f_n|, \phi'g \in \mathfrak{F} \quad \text{we see that} \quad N(\phi'f - \phi'g) \leq \sum \limits_{n=1}^{\infty} N(\phi'|f_n|) = \sum \limits_{n=1}^{\infty} L(\phi'|f_n|) = \sum \limits_{n=1}^{\infty} L'(|f_n|) = \sum \limits_{n=1}^{\infty} E'(|f_n|) \leq \epsilon \quad \text{and hence that} \quad \phi'f \in \mathfrak{F}, |L(\phi'f) - L(\phi'g)| \leq \epsilon.\]

The equation \( L(\phi'g) = L'(g) \) yields \( |L'(f) - L(\phi'f)| \leq 2\epsilon \).

In considering the converse, it is convenient to observe that the relation \( N' \leq N \) requires every null set in the \( E' \)-theory to be a null set in the \( E' \)-theory also. Thus when \( \phi'f \in \mathfrak{F} \) we see that \( f \) is finite almost everywhere in the sense of either theory. Since we can examine separately the two functions \( |f| \) and \( |f| - f \), where \( \phi'|f| = \phi'|f| \in \mathfrak{F}, \phi'(|f| - f) = \phi'|f| - \phi'f \in \mathfrak{F} \) and \( f = |f| - (|f| - f) \), there is no loss of generality in supposing that \( f \geq 0 \).

Then on putting \( g_\epsilon = f/(n\phi' + 1), h_\epsilon = n\phi'/(n\phi' + 1) \) we see that the sequences \( \{g_\epsilon\}, \{h_\epsilon\} \) are respectively monotonically decreasing and increasing almost everywhere to the corresponding limit functions \( g \) and \( h \).

Since \( g_\epsilon + h_\epsilon = f \), if we have \( g + h = f \), almost everywhere; and we verify similarly that \( \phi'g = 0, \phi'h = \phi'f \) almost everywhere. The relation \( h_\epsilon \in \mathfrak{F} \) implies that \( h_\epsilon \in \mathfrak{F}' \) and \( L'(h_\epsilon) = L(\phi'h_\epsilon) \leq L(\phi'f) \). Hence I (12) shows that \( h \in \mathfrak{F}' \). Turning now to a consideration of \( g \), we shall show that \( g \in \mathfrak{F}' \) and \( L'(g) = 0 \). For this purpose, we construct a function \( f_0 \geq 0 \) in \( \mathfrak{F} \) such that \( \{x; f_0(x) = 0\} \) is a null set, in the manner indicated above. Since \( f_0 \in \mathfrak{F}' \) and \( \phi'f_0 \in \mathfrak{F} \) we can express \( f_0 \) also in the form \( g_0 + h_0 \) just described.

We note that \( g_0 = f_0 - h_0 \in \mathfrak{F}', N'(g_0) = L'(g_0) = L(\phi'g_0) = 0 \). Clearly we have \( g_0(x) > 0 \) whenever \( \phi'(x) = 0 \), except possibly on a null set; and hence we also have \( g_0(x) > 0 \) whenever \( g(x) > 0 \), with a like qualification.

The relation \( g \leq g_0 + g_0 + g_0 + \ldots \) therefore holds almost everywhere and implies that \( N'(g) \leq N'(g_0) + N'(g_0) + N'(g_0) + \ldots = 0 \). Thus \( g \) is a null function in the \( E' \)-theory and as such has the properties desired. Since \( g \) and \( h \) are both in \( \mathfrak{F}' \) we have \( f = g + h \in \mathfrak{F}' \). This completes the proof of (3). It is desirable to mention without proof that (3) has the following easy converse:

(4) if \( E \) is given and \( \phi \) is any function in \( \mathfrak{M} \) such that \( 0 \leq \phi \leq 1 \), then the operation \( E' \) defined over \( \mathfrak{F} \) by the equation \( E'(f) = L(\phi f) \) is an elementary integration such that \( E' \leq E \); and the function \( \phi' \) obtained by virtue of (3) differs from \( \phi \) at most on a null set.
Clearly (3) and (4) have the following interpretation: the correspondence $E' \rightarrow \phi'$ is an isomorphism (both algebraic and ordinal) between the totality of $E' \leq E$ and the totality of $\phi' \in M$, $0 \leq \phi' \leq 1$, identified modulo null functions. In particular, the familiar lattice properties of this totality of functions are reflected in the system of all $E' \leq E$. It is then easy to see that the system of all elementary integrations is a lattice with respect to the ordering relation $\leq$; and that in it every monotonely decreasing sequence has a greatest lower bound. The isomorphism leads directly to the complete Radon-Nikodym theorem and to the Lebesgue decomposition theorem, as we shall now show. It is convenient to introduce the following definition of a new order relation $\rightarrow$ between elementary integrations: $E' \rightarrow E^*$ if and only if every null set in the $E^*$-theory is a null set in the $E'$-theory (or, equivalently, if and only if every null function in the $E^*$-theory is a null function in the $E'$-theory). Evidently $E' \leq E^*$ implies $E' \rightarrow E^*$; and when $E = E' + E^*$ the condition on the associated functions $\phi'$ and $\phi^*$ equivalent to the condition $E' \rightarrow E^*$ is that $\phi^*(x) = 0$ imply $\phi'(x) = 0$ for almost all $x$. We can now state the Lebesgue decomposition theorem as follows:

(5) (Lebesgue) if $E'$ and $E^*$ are given elementary integrations over $E$, then $E'$ has a unique decomposition of the form $E' = E'_1 + E'_2$ where $E'_1 \rightarrow E^*$ and $(E'_2 \cap E^*)(|f|) = 0$ for all $f$ in $E$.

The proof is given by putting $E = E' + E^*$ and considering the associated functions $\phi'$ and $\phi^*$. In order to obtain the desired decomposition we evidently have to determine $\phi'_1$ and $\phi'_2$ so that $\phi' = \phi'_1 + \phi'_2$, $\phi^*(x) = 0$ implies $\phi'_1(x) = 0$ for almost all $x$, and $\min (\phi'_2, \phi^*) = 0$ almost everywhere. Then $\phi'_1$ and $\phi'_2$ are essentially unique and can be expressed in terms of $\phi'$ and $\phi^*$ by the formulae $\phi'_1 = \lim_{n \rightarrow \infty} n\phi'/\phi + 1$, $\phi'_2 = \lim_{n \rightarrow \infty} \phi'/n\phi^* + 1$ similar to ones used in the discussion of (3). The complete Radon-Nikodym theorem reads:

(6) (Radon-Nikodym) if $E'$ and $E^*$ are given elementary integrations over $E$ such that $E' \rightarrow E^*$, then there exists an essentially unique non-negative function $\chi$ in $M$ such that $f \in M$ if and only if $\chi f \in M$, while $L'(f) = L''(\chi f)$; in particular $E'(f) = L''(\chi f)$ when $f \in E$. Conversely, if $E''$ is given and if $\chi$ is any non-negative function in $M''$ such that $f \in E'$ implies $\chi f \in M''$, then the operation $E'$ defined by the equation $E'(f) = L''(\chi f)$ is an elementary integration such that $E' \rightarrow E''$.

The proof is again based on the isomorphism established in (3) and (4). For given $E'$ and $E^*$, we put $E = E' + E^*$ and determine the associated functions $\phi'$ and $\phi^*$ in $M \subset M' \cap M''$ in accordance with (3). Since $\phi' + \phi^* = 1$ and $\phi^*(x) = 0$ implies $\phi'(x) = 0$ for almost all $x$, we see that $\phi^*$ vanishes at most on a null set. Consequently the function $\chi = \lim_{n \rightarrow \infty} n\phi'/
\((n\phi'' + 1)\) is equal almost everywhere to \(\phi'/\phi''\); and the equations \(\phi' = \chi/(\chi + 1)\), \(\phi'' = 1/(\chi + 1)\) hold almost everywhere. It is clear that \(\chi \in \mathbb{M} \subset \mathbb{M}'\). From (3) we now conclude that \(f \in \mathcal{E}'\) if and only if \(\phi''(\chi f) = \phi'\) and hence if and only if \(\chi f \in \mathcal{E}'\). When these conditions are verified we have \(L'(f) = L(\phi' f) = L(\phi''(\chi f)) = L''(\chi f)\). On the other hand, when \(E''\) and \(\chi\) are given it is easy to verify that \(E'\) is an elementary integration. We therefore put \(E = E' + E''\) and determine the associated functions \(\phi'\) and \(\phi''\) in \(\mathbb{M}\). If \(f \in \mathcal{E}\) we have \(L(\phi' f) = E'(f) = L''(\chi f) = L(\phi'' f)\). Thus \(\chi\) is finite for almost all \(x\) and the product \(\phi'' \chi\) is defined and non-negative almost everywhere.

We observe now that the equation \(L(\phi' f) = L(\phi'' \chi f)\) can be established by continuity for all \(f \in \mathcal{E}\). For such an \(f\), let \(\{f_n\}\) be a sequence of functions in \(\mathcal{E}\) which converges in \(\mathcal{E}\) to \(f\). We may suppose in addition that \(\lim f_n\) exists almost everywhere and differs from \(f\) only on a null set. Then \(\lim_{n \to \infty} \phi'' \chi f_n = \phi'' \chi f\) almost everywhere. At the same time \(L(\phi'' \chi f_m - \phi'' \chi f_n) = L(\phi'' \chi |f_m - f_n|) = L(\phi' |f_m - f_n|) \leq L(|f_m - f_n|)\) so that \(\{\phi'' \chi f_n\}\) converges in \(\mathcal{E}\) necessarily to \(\phi'' \chi f\). We obviously have \(L(\phi'' \chi f) = \lim_{n \to \infty} L(\phi'' \chi f_n) = \lim_{n \to \infty} L(\phi' f_n) = L(\phi' f)\). By (1) we can construct the function \(f_0\) used in the proof of (3). Since we have \(\phi'' \chi f_0 \in \mathcal{E}\) it follows that \(\phi'' \chi = \lim_{n \to \infty} \phi'' \chi f_0/(n \phi_0 + 1) \in \mathbb{M}\). With further use of (1) we can infer from the fact that \(L(\phi' - \phi'' \chi) f) = 0\) for all \(f \in \mathcal{E}\), \(\phi' - \phi'' \chi \in \mathbb{M}\), that \(\phi' - \phi'' \chi = 0\) almost everywhere. Thus \(\phi''(x) = 0\) implies \(\phi'(x) = 0\) for almost all \(x\), and we must have \(E' \rightarrow E''\). The formulae connecting \(\phi'\), \(\phi''\), \(\chi\) are evidently the same as those noted above, and \(\chi \in \mathbb{M}\). The relations \(E'' \rightarrow E\), \(E \rightarrow E''\) are both valid. We conclude the work of this section by an observation concerning the equivalence relation \(\sim\) defined by putting \(E' \sim E''\) if and only if both \(E' \rightarrow E''\) and \(E'' \rightarrow E'\). We have:

\((7)\) \(with\ respect\ to\ the\ order\ relation\ \rightarrow\ and\ the\ equivalence\ relation\ \sim\ the\ totality\ of\ elementary\ integrations\ defined\ on\ the\ family\ \mathcal{E}\ of\ elementary\ functions\ is\ a\ \sigma\)-additive\ generalized\ Boolean\ algebra.

If we consider the \(E'\) such that \(E' \subseteq E\) and interpret the relations \(\sim\) and \(\rightarrow\) in the isomorphic system of functions \(\phi'\), we see at once that each equivalence class contains exactly one characteristic function (modulo null functions) and that for characteristic functions the relation corresponding to \(\rightarrow\) is the natural ordering for real functions. The system of elementary integrations \(E'\) such that \(E' \subseteq E\) is therefore a \(\sigma\)-additive Boolean algebra, as stated. The remainder of the theorem then follows, provided that the \(\sigma\)-additivity be understood to mean that any \textit{bounded} sequence has a Boolean sum.

Turning now to the second part of our problem—the study of the extensions of a given \(E\)—we see at once that
(8) if $E'$ is an extension of $E$ from $E$ to $E'$, then $N' \leq N$, $\mathcal{F}' \supseteq \mathcal{F}$, $\mathcal{V}' \supseteq \mathcal{V}$, and $L'$ is an extension of $L$ from $L$ to $L'$; if in particular $E \subseteq E' \subseteq E$ then $N' = N$, $\mathcal{F}' = \mathcal{F}$, $\mathcal{V}' = \mathcal{V}$ and $L' = L$.

The first part is evident, the second almost so. However we may remark that the second part results most easily from considering $L$ as an extension $E^*$ of $E$ and $E'$ over $E^*$, since II (14) then immediately gives $N^* = N$ and hence $N \leq N^* \leq N' \leq N, N' = N$. It appears that very little indeed can be said about the general problem of extensions, largely because the condition of I (2) makes the construction of extensions a matter of considerable delicacy. When this condition is dropped it is possible to obtain certain results which are not altogether without interest. We may illustrate this by the following theorem:

(9) the elementary integration $E$ over $E$ has an extension $J$ over $\mathcal{F}$ which is a positive linear functional with the properties $0 \leq J(|f|) \leq N(f)$, $J(f) = L(f)$ when $f \in \mathcal{F}$.

The proof is an immediate consequence of the Hahn-Banach theorem: we put $p(f) = N(f^+)$ where $f^+ = \frac{1}{2}(|f| - f)$, observing that $0 \leq p(f)$, $p(f + g) \leq p(f) + p(g)$, $p(\alpha f) = \alpha p(f)$ when $\alpha \geq 0$, and $L(f) \leq L(f^+) = p(f)$ when $f \in \mathcal{F}$; and we conclude that $L$ has a linear extension $J$ over $\mathcal{F}$ such that $J(f) \leq p(f)$ for all $f$ in $\mathcal{F}$, a condition which implies for any $f \geq 0$ that $J(-f) \leq p(-f) = N((-f)^+) = N(0) = 0$, $J(f) \geq 0$. The relations $0 \leq J(|f|) \leq p(|f|) = N(|f|) = N(f^+) = N(f)$ then hold also.

Having done what little we can with the extension problem, we shall now turn to a discussion of the general integral due to Bourbaki. This integral is defined in a manner quite similar to that followed in these notes; but somewhat more stringent conditions are imposed on the elementary functions and integrals, while more general processes are employed for the construction of the general integral. As we shall see, it follows that whenever the Bourbaki theory is applicable our theory is also and yields a general integration which has the general integration of Bourbaki as an extension, of remarkably simple type: this extension can, in fact, be obtained essentially by augmenting the class of null functions. It is thus apparent that the discussion of the Bourbaki integral is appropriately included in the present note. The definition of the Bourbaki integral rests upon the use of function-systems filtering with respect to the natural ordering for functions: a class of functions is said to be a filtering (or directed) system if whenever it contains $f$ and $g$ it also contains a function $h$ such that $h \geq \max (f, g)$. Here we shall always denote such a system by the letter $\mathcal{R}$. Let there be given a system $\mathcal{E}$ of elementary functions and an elementary in-
tegration $E$ which satisfy the conditions of I (1) and of the following strengthened version of I (2), namely:

(2) if $\mathcal{F}$ is a filtering system in $\mathcal{E}^+$, the system of non-negative functions in $\mathcal{E}$, and if $f$ is any function in $\mathcal{E}$ such that $|f(x)| \leq \sup_{k \in \mathbb{R}} k(x)$, then $E(|f|) \leq \sup_{k \in \mathbb{R}} E(k)$.

A quantity analogous to $N(f)$ can then be introduced through the definition

$$N(f) = \inf_{\mathcal{F} \subset \mathcal{E}^+} \{ \lambda; \lambda = \sup_{k \in \mathbb{R}} E(k), |f(x)| \leq \sup_{k \in \mathbb{R}} k(x) \}$$

If $|f| \leq \sum_{n=1}^{\infty} |f_n|, f_n \in \mathcal{E}$, we can take $\mathcal{F}$ as the totality of partial sums of this infinite series; and we immediately infer that $\bar{N} \leq N$. Moreover properties I (5)–I (9) are readily verified for $\bar{N}$. Hence whatever parts of the theory presented in these notes have been built directly on these properties have valid analogs in the theory of Bourbaki. In particular, we see that $\bar{F} = \{ f; \bar{N}(f) < + \infty \}$ is a Banach space, and we can define $\bar{F}$ as the closure of $\mathcal{E}$ in $\bar{F}$, taking $\bar{L}(f) = \bar{N}(f^+) - \bar{N}(f^-) = \bar{F}(f)$ for $f \in \bar{F}$. A detailed check shows that the only results in our theory which depend upon a direct use of the definition of $N$ are I (5)–I (9), II (10), II (14), III (1) and one observation in the proof of (3) above. Thus it remains for us to prove analogs of the last-mentioned results for the Bourbaki theory. We must observe further that in connection with the Fubini theorem in III and also at certain points in the present note it was necessary to check the validity of I (2) for particular families of functions. At the same points the corresponding check of (2) clearly has to be made in the Bourbaki theory. This can be accomplished in every instance without difficulty. Our program for treating the Bourbaki integral has therefore been reduced to two steps, the first being to prove a result of general interest from which II (14), II (10) and the point referred to in the proof of (3) above, all easily follow, namely:

(10) $f \in \bar{F}$ if and only if $f = g + h$ where $g \in \mathcal{F}$ and $h$ is a null function in the sense that $\bar{N}(h) = 0$, it being possible when $f \geq 0$ to choose $g \geq 0$ and $h \geq 0$ so that $\bar{N}(f) = \bar{N}(g) = \bar{N}(h)$; likewise, $f \in \bar{F}$ if and only if $f = g + h$ where $g \in \mathcal{F}$ and $\bar{N}(h) = 0$—and, when these conditions hold, $\bar{L}(f) = \bar{L}(g) = \bar{L}(h)$.

We first observe that in consequence of the relation $N \leq \bar{N}$ we have $\bar{F} \supset \mathcal{F}$, $\mathcal{F} \supset \mathcal{E}$, and $\bar{L}(f) = \bar{L}(f)$ for all $f \in \mathcal{F}$. Thus $\bar{L}$ is an extension of $L$. The sufficiency of the conditions stated in the theorem is evident. In discussing their necessity, we may treat the case where $f \geq 0$ since the general case can be reduced to it by consideration of the functions $f^+$ and $f^-$. Assuming $f \geq 0$
and \( \bar{N}(f) < +\infty \), we choose a filtering system \( \mathcal{F}_p \subset \mathcal{C}^+ \) so that \( 0 \leq f(x) \leq \sup k(x) \), \( \bar{N}(f) \leq \lambda_p = \sup_{k \in \mathcal{F}_p} E(k) \leq \bar{N}(f) + 1/p \). Evidently this system contains an increasing sequence \( \{k_{np}\} \) such that \( \lambda_p = \lim_{n \to \infty} E(k_{np}) \). In accordance with I (12) this sequence has a limit \( k_p \) in \( \mathcal{F} \). Putting \( g_p = \min (f, k_p) \geq 0 \), we see that \( g_p \in \mathcal{F} \). We also put \( h_p = \max (f, k_p) - k_p \geq 0 \) and show that \( \bar{N}(h_p) = 0 \). The equation \( f = g_p + h_p \) then gives the desired result. In order to estimate \( \bar{N}(h_p) \) we put \( h_{np} = \max (f, k_{np}) - k_{np} \), obtaining a sequence which decreases with \( 1/n \) to the limit \( h_p \). Hence \( \bar{N}(h_p) \leq \bar{N}(h_{np}) \). The functions max \( (k, k_{np}) - k_{np} \geq 0 \), \( k \in \mathcal{F}_p \) constitute a filtering system in \( \mathcal{C}^+ \) and for each \( x \) in \( X \) have a supremum not less than \( h_{np}(x) \geq 0 \). Consequently \( \bar{N}(h_{np}) \leq \sup_{k \in \mathcal{F}_p} (E(\max(k, k_{np}) - E(k_{np})) = \lambda_p - E(k_{np}) \). Passing to the limit we have \( \bar{N}(h_{np}) \to 0 \) and hence \( \bar{N}(h_p) = 0 \). It is clear that \( \bar{N}(g_p) \leq \bar{N}(f) \leq \bar{N}(g_p) + \bar{N}(h_p) \) and hence that \( \bar{N}(g_p) = \bar{N}(f) \). However all we know concerning \( \bar{N}(g_p) \) is that \( \bar{N}(f) = \bar{N}(g_p) \leq \bar{N}(g_p) \leq \bar{N}(k_p) = \lambda_p \leq \bar{N}(f) + 1/p \). To complete the discussion we therefore put \( g = \inf g_p \geq 0 \), noting that \( f \geq g \), \( g \in \mathcal{F} \), \( f = g \) almost everywhere in the Bourbaki theory. Writing \( h = f - g \geq 0 \), we have \( \bar{N}(h) = 0 \). Now \( \bar{N}(f) \leq \bar{N}(g) + \bar{N}(h) = \bar{N}(g) \leq \bar{N}(g) \leq \bar{N}(g_p) \leq \bar{N}(f) + 1/p \) for all \( p \), and \( \bar{N}(f) = \bar{N}(g) = \bar{N}(g), \) as we wished to prove. Assuming now that \( f \) is an arbitrary function in \( \mathcal{F} \) we know that there is a sequence \( \{f_n\} \) in \( \mathcal{C} \) such that \( \bar{N}(f - f_n) = \bar{L}(f - f_n) \to 0 \) while \( \bar{N}(f_n - f_{n+1}) = \bar{L}(|f_n - f_{n+1}|) = E(|f_n - f_{n+1}|) \leq 2^{-n} \). It follows that \( \{f_n\} \) also converges in \( \mathcal{F} \) to a function \( g \). Clearly we must have \( f = g \) almost everywhere in the Bourbaki theory, since both functions are equal almost everywhere to \( \lim \sup f_n \). Thus we have \( f = g + h \) where \( \bar{N}(h) = 0 \). It is easy to see that \( \bar{L}(f) = \bar{L}(g) + \bar{L}(h) = \bar{L}(g) = L(g) \) here. This completes the proof. The second step of our program is to give a direct demonstration of III (1) for the Bourbaki theory. We have

(11) if the elementary integrations \( E_x, E_y \) and \( E_z \) satisfy the relation \( E_z \subset E_x \ast E_y \), then the corresponding operations \( \bar{N}_x, \bar{N}_y, \bar{N}_z \) are such that \( \bar{N}_x f(z) \geq \bar{N}_z \bar{N}_y f(x, y) \) for every \( f \) in \( \mathcal{G}(x) \).

The proof is simple. Since there is nothing to prove unless \( \bar{N}_x f(z) < +\infty \), we assume the latter relation. We can then choose for any given \( \epsilon > 0 \) a filtering system \( \mathcal{F} \subset \mathcal{C}^+ \) such that \( |f(z)| \leq \sup_{k \in \mathcal{F}} k(z) \), \( \bar{N}_x f(z) \leq \sup_{k \in \mathcal{F}} E(k) \leq \bar{N}_x f(z) + \epsilon \). The functions \( k(x, y) \) for fixed \( x \) constitute a filtering system in \( \mathcal{C}^+(Y) \) and their supremum is not less than \( |f(x, y)| \). Hence we must have \( \bar{N}_y f(x, y) \leq \sup_{k \in \mathcal{F}} E_y k(x, y) \). Since the functions \( E_y k(x, y), \ k \in \mathcal{F}, \) constitute a filtering system in \( \mathcal{C}^+(X) \), we must have \( \bar{N}_x \bar{N}_y f(x, y) \leq \sup_{k \in \mathcal{F}} E_x E_y k(x, y) = \sup_{k \in \mathcal{F}} E_x k(z) \leq \bar{N}_x f(z) + \epsilon \). The theorem then follows at
once. We now see that all the results established in our theory hold, mutatis mutandis, for the theory of Bourbaki. The latter theory is calculated to give the best possible handling of the case where $\mathcal{C}$ is the family of all continuous functions with compact nuclei on a locally compact space $X$. We shall therefore conclude our remarks with some comments on this case. In the Bourbaki theory it is not difficult to verify that the Riesz-Markoff-Kakutani theorem, II (18), assumes precisely the form announced by Markoff: the measure is a regular measure such that every compact subset of $X$ is measurable. Thus where our theory produces a Baire measure, the Bourbaki theory produces a Borel measure. When $X$ is the homogeneous space of the left cosets of a closed subgroup $Y$ of a locally compact topological group $Z$, the discussion given in III can be repeated verbatim in the Bourbaki theory. Now, however, the final result is a generalization of Ambrose’s theorem cited there. We thus obtain a simplified proof in which we do not need to assume, as Ambrose does, that $Y$ is a normal subgroup.


2 After the preparation of our third note, we learned that Bourbaki’s projected treatment of the theory of integration had assumed a form remarkably like that described in these notes. The resemblance extends even to some of the finer details. The comparative study outlined in the present note should suffice to clarify the technical relations between the two theories.

3 If such a sequence exists it can be replaced by a similar sequence in $\mathcal{C}$, by a simple application of the definition of $\mathcal{C}$.


6 Stone, M. H., Linear Transformations in Hilbert Space and Their Applications to Analysis, New York, 1932, Theorem 2.28. The proof given there for the complex separable case applies also to the real unrestricted case, which we need here.

7 Riesz, F., “Sur quelques notions fondamentales dans la théorie générale des opérations linéaires,” Ann. Math., 41, 174–206 (1940), has shown how the lattice properties can be obtained directly. Stronger properties than those stated here are actually valid and can be established in general by Riesz’s methods or in this instance by direct examination of the function-lattice.

8 Saks, S., loc. cit.

9 Saks, S., loc. cit.


11 This result is to be compared with one given by Banach, S., Théorie des Opérations Linéaires, Warszawa, 1932, pp. 29–32.