ON n-DIMENSIONAL CONCEPTS OF BOUNDED VARIATION, ABSOLUTE CONTINUITY AND GENERALIZED JACOBIAN

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Background.—A real-valued function \( x = f(u) \) may be thought of as defining a mapping \( f: U^1 \rightarrow X^1 \), where \( U^1, X^1 \) designate Euclidean 1-spaces (real number-lines). In this interpretation the fundamental concepts of bounded variation, absolute continuity and derivative acquire important and suggestive geometrical meaning. To obtain fruitful \( n \)-dimensional generalizations of these fundamental concepts, it is then natural to study mappings \( f: U^n \rightarrow X^n \), where \( U^n, X^n \) are Euclidean \( n \)-spaces with points \( u = (u_1, \ldots, u_n), x = (x_1, \ldots, x_n) \). The general objective is to develop concepts of bounded variation, absolute continuity and generalized Jacobian, which yield a theory comparable in utility and scope with the classical one-dimensional theory. Various approaches to this general objective may be found in the literature. The present note is concerned with the line of thought presented in reference 1. The theory in reference 1 is developed for the case \( n = 2 \). For this case, the theory may be considered as essentially complete, due to the sustained efforts of many mathematicians. The generalization to the \( n \)-dimensional case represents a long-range program of great interest and great difficulty. The purpose of this note is to report on progress achieved by the writers in this general direction. We shall present in detail the precise definitions of the \( n \)-dimensional concepts of bounded variation, absolute continuity and generalized Jacobian which we use in our work, and we shall describe applications to the transformation of multiple integrals, a topic which is discussed in Part IV of reference 1 for the case \( n = 2 \).

Basic Concepts.—Let \( D_0 \) be a bounded domain (connected open set) in the space \( U^n \), and let \( f: D_0 \rightarrow X^n \) be a bounded continuous mapping from \( D_0 \) into the space \( X^n \). In analogy with the 2-dimensional case it is expedient to assign to every point \( x \) of \( X^n \) an essential multiplicity relative to the mapping \( f \). For the 2-dimensional case, various equivalent definitions of the essential multiplicity are available—in fact, four such essential multiplicities were actually used in the literature. In reference 2 these essential multiplicities are denoted by \( \psi, \psi^*, \Psi, \Psi^* \), respectively. Since they differ from each other only on countable sets, they can be used interchangeably throughout the 2-dimensional theory. However, the corresponding essential multiplicities in \( n \) dimensions may differ from each other on sets of positive measure. Recent results of Federer\(^3\) indicate that the multiplicity corresponding to the largest one of the four multi-
plicities just mentioned, namely $\Psi^*$, is the appropriate one for the $n$-dimensional case. In the work of Federer, the definition of this multiplicity is stated, in a very convenient manner, in terms of relative Cech cohomology groups. This essential multiplicity is used in the present note, and is denoted by $K$. For the concept of absolute continuity several equivalent definitions are available in the 2-dimensional case. For the $n$-dimensional case the choice is again restricted; we found it appropriate to adopt the definition proposed in reference 4. Finally, for the $n$-dimensional generalized Jacobian we found it convenient to adopt the form of the definition suggested in reference 5 for the 2-dimensional case. Precise definitions of these basic concepts follow.

**The Index** $\mu(x, f, D).$—Given a bounded continuous mapping $T: D_0 \to X^n$ as above, let $x_0$ be any point in $X^n$. A domain $D$ is termed admissible for $x_0, f, D_0$ if the closure $cD$ of $D$ is contained in $D_0$ and $x_0$ is not in the image of the frontier $frD$ of $D$ under $f$. If $D$ is admissible for $x_0, f, D_0$, there exist in $X^n$ pairs of compact sets $A, B$ satisfying the following conditions: (i) $frA$ is a subset of $B$, and $B$ is a subset of $A$; (ii) $A - B$ is not empty and is connected; (iii) the image of $cD$ under $f$ is contained in $A$; (iv) the image of $frD$ under $f$ is contained in $B$; (v) $x_0$ is in $A - B$. Under these conditions the pair $A, B$ is termed admissible for $x_0, f, D$. Assume that $A, B$ is admissible for $x_0, f, D$, and consider the mapping $f|cD: (cD, frD) \to (A, B)$. This mapping induces a homomorphism $h: H^n(A, B) \to H^n(cD, frD)$, where $H^n$ denotes the $n$-dimensional Cech cohomology group with integral coefficients. Both of the groups involved are infinite cyclic, and by a suitable orientation process one may select once for all standard generators $g_{cD}, g_{AB}$ for the groups $H^n(cD, frD), H^n(A, B)$, respectively. Then $h(g_{AB}) = k \cdot g_{cD}$, where $k$ is an integer (positive, negative or zero). This integer $k$ is defined to be the index $\mu(x_0, f, D)$; it can be easily shown to depend solely upon $x_0, f, D$ as the notation suggests. If $\mu(x_0, f, D)$ is not zero, then $D$ is termed an indicator domain for $x_0, f, D_0$. An indicator domain $D$ is termed positive (negative) if the sign of $\mu(x_0, f, D)$ is positive (negative).

**The Essential Multiplicity Function** $K(x, f, D_0).$—Let $\sigma$ be a generic notation for a finite system of pairwise disjoint indicator domains $D$ for $x_0, f, D_0$. Then $K(x_0, f, D_0)$ is defined to be the least upper bound of $\sum |\mu(x_0, f, D)|$ for $D$ in $\sigma$, with respect to all systems $\sigma$. If each $D$ occurring in the system $\sigma$ is restricted to be a positive (negative) indicator domain, one obtains the definition of the multiplicity functions $K^+(x_0, f, D_0)$, $K^-(x_0, f, D_0)$.

**The Essential Maximal Model Continuum.**—A continuum $c$ is termed an essential maximal model continuum for $x_0, f, D_0$ if $c$ is a component of the inverse of $x_0$ under $f$ in $D_0$, and every neighborhood of $c$ contains in its interior an indicator domain $D$ which contains $c$. The essential set
$E(f, D_0)$ consists of all points $u_0$ in $D_0$ which belong to an essential maximal model continuum for $f(u_0), f, D_0$.

**Essential Bounded Variation, Essential Absolute Continuity.**—The mapping $f$ is said to be eBV in $D_0$ (essentially of bounded variation in $D_0$) if $K(x, f, D_0)$ is summable. The mapping $f$ is said to be eAC in $D_0$ (essentially absolutely continuous in $D_0$) if (i) it is eBV in $D_0$, and (ii) for every subset $b$ of $E(f, D_0)$ which has measure zero, it is true that the measure of the image of $b$ under $f$ is also zero.

The Signed Essential Multiplicity Function $\mu_e(x, f, D_0).$—This is defined to be $K^+(x, f, D_0) - K^-(x, f, D_0)$ unless both $K^+$ and $K^-$ are infinite. When $f$ is eBV in $D_0$, $\mu_e(x, f, D_0)$ is defined almost everywhere in $X^n$ and is summable. If $D$ is an admissible domain for $x_0, f, D_0$, and $K(x_0, f, D)$ is finite, then the index $\mu(x_0, f, D)$ and the signed essential multiplicity $\mu_e(x_0, f, D)$ have the same value.

The Essential Generalized Jacobian $J_e(u, T).$—Assume that $f$ is eBV in $D_0$. For each open interval $I$ in $D_0$, the signed essential multiplicity function $\mu_e(x, f, I)$ is defined almost everywhere in $X^n$ and is summable. The Lebesgue integral of $\mu_e(x, f, I)$ yields a function of intervals $I$ in $D_0$ which possesses a derivative almost everywhere in $D_0$; this derivative is the essential generalized Jacobian $J_e(u, f)$.

**Application to the Transformation of n-tuple Integrals.**—In terms of the preceding concepts, transformation formulae for $n$-tuple integrals are established which represent complete generalizations of the 2-dimensional formulae in reference 1, IV. 4.4, IV. 4.6, IV. 4.7, IV. 4.9, with the signed essential multiplicity $\mu_e$ replacing the function $\nu$ in these formulae. As a generalization of reference 1, IV. 4.8, it is shown that if $f$ is eBV in $D_0$, then the integral over $D_0$ of $|J_e(u, f)|$ is less than or equal to the integral over $X^n$ of $K(x, f, D_0)$, the sign of equality holding if and only if $f$ is eAC in $D_0$. The range of applicability of these $n$-dimensional transformation formulae depends upon the scope of the eAC mappings. It is shown that this class is closed under various limit processes, the results representing complete generalizations of the closure theorems in reference 1, IV. 4.11, IV. 4.12, IV. 4.15, IV. 4.16, IV. 4.17, IV. 4.18, IV. 4.19. It is demonstrated that if $f$ is eBV in $D_0$ and if the coordinate functions defining $f$ possess complete differentials almost everywhere in $D_0$, then $J_e(u, f)$ agrees with the ordinary Jacobian almost everywhere in $D_0$. Consequently the $n$-dimensional transformation formulae contain, as very special cases, various special results established in previous literature. Many further results presented in Part IV of reference 1 admit of a complete generalization to $n$ dimensions.

A detailed presentation of our work is now being prepared for publication.

QUADRATIC RELATIONS INVOLVING THE NUMBERS OF SOLUTIONS OF CERTAIN TYPES OF EQUATIONS IN A FINITE FIELD

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In two previous papers¹ the writer considered the problem of finding the number of sets \( s, t \), such that

\[ g^i + m_1t + g^j + m_2t + 1 = 0, \tag{1} \]

where \( m_1 \) and \( m_2 \) are integers \( 0 < m_1 < p^n - 1; \) \( 0 < m_2 < p^n - 1; \) \( p^n - 1 \equiv 0 \pmod{m_1}, \) \( p^n - 1 \equiv 0 \pmod{m_2}, \) \( g \) is a primitive root in a finite field \( F(p^n) \), \( p \) an odd prime, \( i \) a given integer in the set \( 0, 1, \ldots, m_1 - 1; \) \( j \) a given integer in the set \( 0, 1, \ldots, m_2 - 1, s \) in the set \( 0, 1, \ldots, m_1' - 1; \) \( t \) in the set \( 0, 1, \ldots, m_2' - 1; \) \( p^n - 1 = m_1m_1' = m_2m_2'. \) Denoting the number of possible sets \( s, t \) by \( (i, j)_{m_1m_2} \), an explicit expression in terms of \( p \) and \( n \) for the expression

\[ \sum_{i = 0}^{m_1 - 1} \sum_{j = 0}^{m_2 - 1} (i, j)_{m_1m_2}(i + h, j + k)_{m_1m_2}, \]

in the two Cases I. \( m_1 = m_2, \) II. \( m_1m_2 = p^n - 1, (m_1, m_2) = 1, \) was set up. In the present paper we obtain, using much simpler methods, similar results for \( m_1 \) and \( m_2 \) any divisors of \( p^n - 1, \) also quadratic relations of a different type when \( m_1 = m_2. \) In Hua and Vandiver² an expression for the number of solutions of (1) was derived, noting that the equation

\[ c_1x_1^{m_1} + c_2x_2^{m_2} + c_3 = 0 \tag{2} \]

in \( F(p^n) \) may be put in the form (1), when \( c_1c_2c_3x_1x_2 \not\equiv 0, \) but the use of this form³ is inconvenient for our present purpose. We have recourse to some results in another paper by the writer.⁴

If we keep \( i \) and \( j \) fixed the number of solutions of (1) is \( m_1m_2(i, j) \) where \( (i, j) \) is the number of different sets \( s, t \) which are possible in the relation

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