THE HOMOTOPY GROUPS OF A TRIAD

BY A. L. BLAKERS AND WILLIAM S. MASSEY

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated by S. Lefschetz, April 2, 1949

In this paper we define new homotopy groups for topological spaces. These groups generalize the homotopy groups of Hurewicz. By the use of these groups and by improved methods we obtain new results about the ordinary homotopy groups, and also easier proofs of known results. Among other things, we can show that \( \pi_6(S^3) \) is non-trivial.

1. One of the principal problems of modern topology is to devise methods for computing the homotopy groups of a space. The homotopy groups of even such simple spaces as spheres have not been computed except in special cases. This contrasts strongly with the situation for the homology groups, which can be computed for any triangulable space. The homotopy groups resemble homology groups in all basic properties except one: the homology groups are invariant under an excision,\(^1\) while this is not generally true for the homotopy groups. More precisely, if a space is a union \( A \cup B \) of two subspaces, then under fairly general conditions the inclusion map \( i: (A, A \cap B) \to (A \cup B, B) \) (called an excision) induces isomorphisms of the corresponding relative homology groups, in all dimensions, but will not generally do so for the relative homotopy groups. This is perhaps the chief reason for the difficulty of computing the homotopy groups. The new homotopy groups defined in this paper are a measure of the deviation from invariance under excision for the relative homotopy groups.

We shall use the following notation for certain subsets of cartesian \( n \)-space, \( C^n \). The coordinates of a point \( x \in C^n \) are denoted by \( (x_1, \ldots, x_n) \), and \( |x| = (x_1^2 + \ldots + x_n^2)^{1/2} \).

\[
\begin{align*}
E^n &= \{ x \in C^n \mid |x| \leq 1 \}, \\
E^{n-1} &= \{ x \in E^n \mid |x| = 1, x_n \leq 0 \}, \\
E^n_+ &= \{ x \in E^n \mid |x| = 1, x_n \geq 0 \}, \\
S^{n-1} &= E^{n-1}_- \cup E^{n-1}_+, \quad S^{n-2} = E^{n-1} \cap E^{n-1}_+, \\
E^n_1 &= \{ x \in E^n \mid x_2 \geq 0 \}, \quad E^n_2 = \{ x \in E^n \mid x_2 \leq 0 \}, \\
\mathcal{P}_0 &= (1, 0, \ldots, 0).
\end{align*}
\]
Let \((X; A, B)\) be a triad; that is, \(A\) and \(B\) are subspaces of the topological space \(X\), and \(A \cap B \neq \emptyset\). Choose a base point \(x_0 \in A \cap B\). A map
\[
f: (E^n; E^n_+, E^n_-, p_0) \rightarrow (X; A, B, x_0)
\]
is a continuous function \(f: E^n \rightarrow X\) such that \(f(E^n_+) \subset A\), \(f(E^n_-) \subset B\), and \(f(p_0) = x_0\). Two such maps \(f_0, f_1\), are homotopic if they are connected by a continuous 1-parameter family of maps,
\[
f_t: (E^n; E^n_+, E^n_-, p_0) \rightarrow (X; A, B, x_0),
\]
where \(0 \leq t \leq 1\).

**Lemma.** If \(n \geq 3\), and \(f: (E^n; E^n_+, E^n_-, p_0) \rightarrow (X; A, B, x_0)\), then there exist homotopic maps \(f', f''\), with \(f'(E^n_+) = f''(E^n_-) = x_0\).

Denote the set of all homotopy classes of mappings \((E^n; E^n_+, E^n_-, p_0) \rightarrow (X; A, B, x_0)\) by \(\pi_n(X; A, B, x_0)\). For \(n \geq 3\), we define an addition between any two elements \(\alpha, \beta \in \pi_n(X; A, B, x_0)\) as follows: Choose maps \(f, g\) belonging to the homotopy classes \(\alpha, \beta\), respectively, with \(f(E^n_+) = g(E^n_+) = x_0\). Define \(h: (E^n; E^n_+, E^n_-, p_0) \rightarrow (X; A, B, x_0)\) by \(h|E^n_+ = f|E^n_+\) and \(h|E^n_- = g|E^n_-\). Then \(\alpha + \beta\) is defined to be the homotopy class of \(h\). With this definition \(\pi_n(X; A, B, x_0)\) becomes a group, called the \(n\)th homotopy group of the triad \((X; A, B)\) at the base point \(x_0\).

Just as in the case of the relative homotopy groups, a continuous map \(f: (X; A, B, x_0) \rightarrow (X'; A', B', x_0')\) induces homomorphisms of the triad homotopy groups. We denote these by \(f_*: \pi_n(X; A, B, x_0) \rightarrow \pi_n(X'; A', B', x_0')\).

Associated with the triad \((X; A, B)\) are two boundary homomorphisms,
\[
\partial: \pi_n(X; A, B, x_0) \rightarrow \pi_{n-1}(A, A \cap B, x_0),
\partial': \pi_n(X; A, B, x_0) \rightarrow \pi_{n-1}(B, A \cap B, x_0),
\]
defined as follows. If \(f: (E^n; E^n_+, E^n_-, p_0) \rightarrow (X; A, B, x_0)\) represents \(\alpha \in \pi_n(X; A, B, x_0)\), \((n \geq 3)\), then \(f|E^n_+\) represents \(\partial(\alpha)\), and \(f|E^n_-\) represents \(\partial'(\alpha)\). Now consider the following sequence of groups and homomorphisms:
\[
\ldots \rightarrow \pi_n(A, A \cap B) \rightarrow \pi_n(X, B) \rightarrow \pi_n(X; A, B) \rightarrow \pi_{n-1}(A, A \cap B) \rightarrow \ldots
\]
The homomorphisms \(i_*\) and \(j_*\) are induced by the inclusion maps
\[
i: (A, A \cap B) \rightarrow (X, B), \quad j: (X; x_0, B) \rightarrow (X; A, B).
\]
This sequence is one of the two homotopy sequences of the triad \((X; A, B)\); the other sequence is obtained by interchanging the roles of \(A\) and \(B\) throughout.

**Theorem 1.** The homotopy sequences of a triad are both exact.

If \(X = A \cup B\), then the inclusion map \(i\) is an excision. It follows from
exactness that $i_*$ is an isomorphism in all dimensions if and only if $\pi_n(A \cup B; A, B) = 0$ for all $n$. Thus if $\pi_n(A \cup B; A, B) \neq 0$ for some integer $n$, then the excision $i$ cannot induce isomorphisms onto in all dimensions, and the groups $\pi_n(A \cup B; A, B)$ are a "measure" of the amount of deviation from invariance under excision.

As an example of the application of these results, consider the triad $(S^n; E_\uparrow, E_\downarrow)$. It can be shown (see below) that $\pi_p(S^n; E_\uparrow, E_\downarrow) = 0$ for $p < 2n - 1$, and that $\pi_{2n-1}(S^n; E_\uparrow, E_\downarrow)$ is infinite cyclic. In this case the excision homomorphism $i_*: \pi_p(E_\uparrow, S^n; E_\downarrow) \rightarrow \pi_p(S^n, E_\downarrow)$ is equivalent to Freudenthal's "Einhangung"; more precisely, in the following diagram,

$$
\begin{array}{c}
\pi_{p-1}(S^n) \\
\pi_p(E_\uparrow, S^n; E_\downarrow) \\
\pi_p(S^n, E_\downarrow)
\end{array}
$$

the boundary homomorphism $\delta$, and the homomorphism $k_*$ induced by an inclusion map, are isomorphisms onto, and the commutativity relation $i_* = k_* \delta$ holds, where $E$ is the Einhangung. These facts, together with the exactness of the homotopy sequence of the triad, imply the Freudenthal theorems.

2. The homotopy groups of triads have many properties which are the analogs of familiar properties of the homotopy groups of spaces or pairs of spaces. We now list some of these properties:

(a) $\pi_n(X; A, B, x_0)$ is abelian for $n > 3$; simple examples show that it need not be so for $n = 3$.

(b) The system of groups $\pi_n(X; A, B, x)$ for $x \in A \cap B$ forms a local system of groups in the space $A \cap B$ in the usual sense, and $\pi_1(A \cap B, x_0)$ is a group of operators on $\pi_n(X; A, B, x_0)$.

(c) If $A \supset B$, then $\pi_n(X; A, B)$ is isomorphic to $\pi_n(X, A)$, and the homotopy sequence of the triad $(X; A, B)$ reduces to that of the triple $(X, A, B)$.

(d) In case $\pi_3(X; A, B)$ is non-abelian, the kernels of the boundary homomorphisms $\delta: \pi_3(X; A, B) \rightarrow \pi_2(A, A \cap B)$ and $\partial': \pi_3(X; A, B) \rightarrow \pi_2(B, A \cap B)$ are both contained in the center of $\pi_3(X; A, B)$.

3. A topological space $X$ is said to be $n$-connected ($n > 0$) if it is arcwise connected and $\pi_p(X) = 0$ for $1 \leq p \leq n$. A pair $(X, A)$ is said to be $n$-connected ($n \geq 1$) if both $X$ and $A$ are arcwise connected, the natural homomorphism $\pi_1(A) \rightarrow \pi_1(X)$ is a homomorphism onto, and $\pi_p(X, A) = 0$, $2 \leq p \leq n$. Similarly, we will say a triad $(X; A, B)$ is $n$-connected ($n \geq 2$) if both of the pairs $(A, A \cap B)$ and $(B, A \cap B)$ are 1-connected, and $\pi_p(X; A, B) = 0$, $2 \leq p \leq n$. By $\pi_2(X; A, B) = 0$ we mean that this set of homotopy classes of mappings contains a single element, the homotopy class of the constant map into $x_0$. 
Let \((X^*, X)\) be a pair, and \(\psi: (E^n, S^{n-1}) \to (X^*, X)\), \((n > 1)\), a continuous map which is a homeomorphism of \((E^n - S^{n-1})\) onto \(X^* - X\). Let \(\hat{E}^n = \psi(E^n)\), \(\hat{E}^n = \hat{E}^n \cap X = \psi(S^{n-1})\). We say that the space \(X^*\) is obtained from \(X\) by adjunction of the cell \(E^n\). Note that \((X^*; X, E^n)\) is a triad, and \(\hat{E}^n\) is 0-connected.

**Theorem 2.** If the pair \((X, \hat{E}^n)\) is \(m\)-connected, \(m \geq 1\), then the triad \((X^*; X, E^n)\) is \((m + n - 1)\)-connected. (In case \(m = 1\), it is also necessary to assume that \(\pi_1(X, \hat{E}^n)\) is abelian; in case \(n = 2\), it is necessary to assume that \(X\) is simple relative to \(\hat{E}^n\) in all dimensions.)

**Theorem 3.** If the space \(\hat{E}^n\) is \(m\)-connected, \(m \geq 1\), then the boundary homomorphism

\[
\delta: \pi_{i+1}(X^*; X, E^n) \to \pi_i(\hat{E}^n, \hat{E}^n)
\]

is the zero homomorphism for \(2 \leq i \leq m + n - 1\).

The proofs of theorems 2 and 3 are a straightforward application of the theory of "obstructions" to extensions and deformations of continuous mappings. By combining these theorems with the exactness of the homotopy sequence of the triad \((X^*; X, E^n)\), important results are obtained as corollaries. Among these are the previously mentioned assertion that \(\pi_p(S^n; E^n_+, E^n_-) = 0\) for \(p < 2n - 1\), and an essential generalization of a theorem of J. H. C. Whitehead\(^6\) about the homotopy groups of the pair \((X^*; X)\).

4. Let \(f: (E^n; E^n_+^1, E^n_-^1) \to (X; A, B)\) be a map, and assume \(u \in H^p(X, A)\), \(v \in H^q(X, B)\), \(p + q = n + 1\) and \(u - v = 0\), where \(u - v\) is the Alexander-Cech-Whitney cup product. In a recent paper,\(^6\) Steenrod has shown how to associate with the elements \(u, v\) an element \(u \triangleright v\) of the factor group \(H^n(E^n, S^{n-1})/f^*H^n(X, A \cup B)\). He has proved that the pairing thus defined, called the functional cup product is bilinear, and depends only on the homotopy class of \(f\). Hence if \(\alpha \in \pi_n(X; A, B, x_0)\) is the homotopy class of \(f\), we can write \(u \triangleright v = u \triangleright v\) without ambiguity.

**Theorem 4.** If \(H^n(X, A \cup B) = H^{n+1}(X, A \cup B) = 0\), and \(\gamma = \alpha + \beta\), \(\alpha, \beta \in \pi_n(X; A, B, x_0)\), \(n > 2\), then

\[
u \triangleright v + u \triangleright v = u \triangleright v.
\]

This conclusion can be alternately stated: The functional cup product determines a homomorphism of the group \(\pi_n(X; A, B)\) into the group of bilinear functions

\[
[H^p(X, A), H^q(X, B)] \to H^n(E^n, S^{n-1}).
\]

In the important special case where \((X; A, B) = (S^n; E^n_+, E^n_-)\), and we use cohomology groups with integer coefficients, additional results can be obtained. It is easily seen that the group of bilinear maps
is an infinite cyclic group. Denote this group by $G_n$. Then we can prove

**Theorem 5.** The homomorphism $\pi_{2n-1}(S^n; E^+, E^-) \rightarrow G_n$ determined by the functional cup product is an isomorphism onto.

Now consider the following portion of the homotopy sequence of the triad $(S^n; E^+, E^-)$:

$$\pi_{2n-1}(S^n; E^+, E^-) \rightarrow \pi_{2n-2}(E^+, S^{n-1}) \rightarrow \pi_{2n-3}(S^n; E^-).$$

Using the known properties of the functional cup product, additional information can be obtained about the homomorphisms $j_*$, $\partial$, and $i_*$. 

(a) If $n$ is odd, then $j_*$ is an isomorphism into, and image $\partial$ is an infinite cyclic subgroup of $\pi_{2n-3}(S^n; E^n, S^n)$. 

(b) If $n$ is even, and $\pi_{2n-1}(S^n)$ has an element of Hopf Invariant 1, then $j_*$ is onto, $\partial = 0$, and $i_*$ is an isomorphism onto. This case is known to occur for $n \equiv 2 \mod 4$, $n > 2$.

(c) If $n$ is even and $\pi_{2n-1}(S^n)$ has no element of Hopf Invariant 1, then it must have an element of Hopf Invariant 2. In this case $j_*$ is a homomorphism onto the subgroup of $\pi_{2n-3}(E^+, S^{n-1})$ of index two, and image $\partial$ is a cyclic subgroup of $\pi_{2n-3}(E^+, S^{n-1})$, of order two. G. W. Whitehead has recently announced that this case occurs when $n = 2, 4, 8$.

These three statements are essentially the Freudenthal theorems, together with some improvements due to G. W. Whitehead.  

5. Let $A$ and $B$ be arcwise connected spaces, $a_0 \in A$, $b_0 \in B$, and let $A \vee B$ be the subspace of the cartesian product $A \times B$ defined by $A \vee B = (A \times b_0) \cup (a_0 \times B)$. $A \vee B$ is the union of $A$ and $B$ with a single point in common. Define $\mu_1: A \rightarrow A \vee B$ by $\mu_1(x) = (x, b_0)$ and $\mu_2: B \rightarrow A \vee B$ by $\mu_2(y) = (a_0, y)$, for $x \in A$, $y \in B$. Consider the following homomorphisms ($n > 1$):

$$\mu_{1*}: \pi_n(A) \rightarrow \pi_n(A \vee B),$$
$$\mu_{2*}: \pi_n(B) \rightarrow \pi_n(A \vee B),$$
$$\partial: \pi_{n+1}(A \times B, A \vee B) \rightarrow \pi_n(A \vee B).$$

We can paraphrase a result of G. W. Whitehead as follows: The homomorphisms $\mu_{1*}$, $\mu_{2*}$, and $\partial$ are isomorphisms into, and $\pi_n(A \vee B)$ splits up into the direct sum

$$\mu_{1*}\pi_n(A) + \mu_{2*}\pi_n(B) + \partial\pi_{n+1}(A \times B, A \vee B).$$

From this it follows by an easy argument using exact sequences, that for $n > 2$, $\pi_n(A \vee B; A, B)$ is isomorphic to $\pi_{n+1}(A \times B, A \vee B)$, and that in the homotopy sequence of the triad $(A \vee B; A, B)$, the boundary operators are both trivial in all dimensions. If $A$ and $B$ are spheres, further
results can be obtained. If theorems 2 and 3 are applied with \( X^* = S^p \times S^q \), and \( E^* \) a \((p + q)\)-simplex in a triangulation of \( S^p \times S^q \), we find that \( \pi_*(S^p \times S^q, S^p \vee S^q) \approx \pi_*(E^{p+q}, S^p \vee S^q) \) for \( i \leq p + q + \min(p, q) - 2 \).

It follows that \( \pi_*(S^p \vee S^q, S^p, S^q) \approx \pi_*(S^{p+q}, S^p, S^q) \) for \( j \leq p + q + \min(p, q) - 3 \).

Let \( \varphi_n: (S^n; E^n_+, E^n_-) \to (S^n \vee S^n_1; S^n_1, S^n_2) \) be the map defined by identifying all points of the "equator" \( S^{n-1} \subset S^n \) to a single point. Then \( \varphi_n \) induces a homomorphism of the corresponding triad homotopy sequences, as shown in the following diagram:

\[
\begin{array}{c}
\cdots \to \pi_*(E^n_+, S^{n-1}) \xrightarrow{i_*} \pi_*(S^n, E^n_-) \xrightarrow{j_*} \pi_*(S^n; E^n_+, E^n_-) \xrightarrow{\varphi_n} \pi_*(S^n \vee S^n_1; S^n_1, S^n_2) \to \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\cdots \to \pi_*(S^n_1) \to \pi_*(S^n_1 \vee S^n_2; S^n_2) \to \pi_*(S^n \vee S^n_1; S^n_1, S^n_2) \to \cdots
\end{array}
\]

Now \( \pi_*(S^n) \approx \pi_*(S^n, E^n) \), and if \( r \leq 3n - 3 \), \( \pi_*(S^{2n-1}) \approx \pi_*(S^n_1 \vee S^n_2; S^n_1, S^n_2) \) by the preceding paragraph. The composition of these two isomorphisms with the homomorphism \( \varphi_n j_* \) of the diagram yields a homomorphism

\[ H: \pi_*(S^n) \to \pi_*(S^{2n-1}, (r \leq 3n - 3). \]

It may be shown that this homomorphism is the same as the Generalized Hopf Homomorphism defined by G. W. Whitehead.\(^7\) However, this definition is simpler and easier to work with than the original definition. Furthermore, we have demonstrated the existence of \( H \) for \( r \leq 3n - 3 \), instead of for \( r \leq 3n - 4 \), as was originally done by Whitehead. By using Whitehead's methods, we can show that in the limiting case for \( n = 3 \), the homomorphism

\[ H: \pi_6(S^3) \to \pi_6(S^3) \]

is a homomorphism onto. Since \( \pi_6(S^3) \) is known to be cyclic of order two, this shows that \( \pi_6(S^3) \neq 0 \). An essential map of \( S^3 \) onto \( S^3 \) may be constructed as follows: Let \( f: S^3 \times S^2 \to S^2 \) be a map of type\(^8\) \((\alpha, \iota)\), where \( \alpha \) is a generator of \( \pi_3(S^2) \) and \( \iota \in \pi_2(S^3) \) is the class of the identity map. Now apply the Hopf construction\(^9\) to \( f \) to obtain a map \( F: S^8 \to S^8 \).

\(^{1}\) Eilenberg, S., and Steenrod, N. E., these PROCEEDINGS, 31, 117–120 (1945). Axiom 6 of this note is called the "excision axiom."


\(^{6}\) Steenrod, N. E., these PROCEEDINGS, 33, 124–128 (1947). A complete exposition will appear soon in the *Ann. Math.*.
ON THE "CENTRAL" PROBABILITY PROBLEM*

by Michel Loève

Department of Mathematics, University of California at Berkeley

Communicated by G. C. Evans, April 4, 1949

Let $\mathcal{L}(X)$, $\mathcal{L}(Y)$ be the probability laws of the real random variables (r. v.) $X$, $Y$, laws characterized either by the distribution functions (d. f.) $F(x)$, $G(x)$ or by their Fourier-Stieljes transforms—characteristic functions (c. f.) $f(u)$, $g(u)$. When $X$ or $Y$ possess subscripts, their d. f. and c. f. will have the same subscripts.

1. The Problem.—The problem of convergence towards a normal law of $\mathcal{L}\left(\sum_{k=1}^{n} X_k/c_n\right)$ when $n \to \infty$, $X_k$'s being independent r. v. and $c_n \to \infty$ being "sur" numbers, has played a central rôle in probability theory. It has been finally solved in 1934–1935 by Feller\textsuperscript{4} and by P. Lévy\textsuperscript{8}.

In its present formulation the "central" problem is that of limit laws, when $n \to \infty$, of sums

$$(\phi) \quad X_n = X_{n,1} + X_{n,2} + \ldots + X_{n,r_n}, \quad r_n \to \infty,$$

under the assumption of independence, i.e.

$$(\varphi) \quad \text{for every fixed } n, \quad X_{n,1}, X_{n,2}, \ldots, X_{n,r_n} \text{ are independent, and of asymptotically and uniformly negligible components:}$$

$$(\mathcal{C}) \quad \max_{1 \leq k \leq r_n} \ P(|X_{n,k}| > \epsilon) \to 0 \text{ for every given } \epsilon > 0.$$

It has been shown that the class of limit laws coincides with that of infinitely divisible (i. d.) laws (Khintchine\textsuperscript{7} also Lévy\textsuperscript{8} after partial results of Bawly\textsuperscript{1}) characterized, roughly speaking, as convolutions of a normal law and a continuum of Poisson laws (P. Lévy\textsuperscript{8} after partial results of Kolmogoroff).\textsuperscript{6} Conditions for convergence towards any i. d. law were given by Gnedenko\textsuperscript{5} and Doeblin.\textsuperscript{3}

The assumption $(\mathcal{C})$ is natural if one wants the limit laws to depend primarily upon the fact that the number $r_n$ of components of sums $X_n$ would be infinitely increasing. But the assumption $(\varphi)$ is very restrictive. When $(\varphi)$ is not assumed, the fundamental result for the primitive form of the problem is due to S. Bernstein:\textsuperscript{2} if $E|X_k|^k < \infty$, normal con-